STEP III, 2016 , Q6 MS

6. Using $R\cosh(x+\gamma) = R(\cosh x\cosh\gamma + \sinh x\sinh\gamma)$, $R = \sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$ if $B > A > 0$. If $B = A$, then $A\sinh x + B\cosh x = Ae^x$. If $-A < B < A$, the expression can be written as $R\sinh(x+\gamma)$ with $R = \sqrt{A^2 - B^2}$ and $\gamma = \tanh^{-1}\frac{B}{A}$. If $B = -A$, then $A\sinh x + B\cosh x = -Ae^{-x}$, and if $B < -A$, the expression can be written as $R\cosh(x+\gamma)$ with $R = -\sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$. For part (i), solving simultaneously gives $a\sinh x + b\cosh x = 1$, which gives the desired solutions using the first result of the question. Similarly for part (ii) using the appropriate result, $x = \sinh^{-1}\left(\frac{1}{\sqrt{a^2-b^2}}\right) - \tanh^{-1}\frac{b}{a}$. For (iii), we require that the conditions for (i) give two solutions, i.e. that $b > a$ and $\left(\frac{1}{\sqrt{b^2-a^2}}\right) > 1$, and so $a < b < \sqrt{a^2+1}$, and vice versa, if this applies there are indeed two solutions. For (iv), we require case (i) to give coincident solutions, i.e. $b = \sqrt{a^2+1}$ and hence $x = -\tanh^{-1}\frac{a}{\sqrt{a^2+1}}$, and so $y = \frac{1}{\sqrt{a^2+1}}$. The reverse argument also applies.

6.

$$R\cosh(x+\gamma) = R(\cosh x\cosh\gamma + \sinh x\sinh\gamma)$$

So we require $A = R\sinh\gamma$ and $B = R\cosh\gamma$ which is possible if $B > A > 0$

Thus $R = \sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$. **B1**

If $B = A$, then $A\sinh x + B\cosh x = Ae^x$ **B1**

If $-A < B < A$, then $A\sinh x + B\cosh x$ can be written

$R\sinh(x+\gamma) = R(\sinh x\cosh\gamma + \cosh x\sinh\gamma)$

requiring $A = R\cosh\gamma$ and $B = R\sinh\gamma$.

So $R = \sqrt{A^2 - B^2}$ and $\gamma = \tanh^{-1}\frac{B}{A}$ **B1**

If $B = -A$, then $A\sinh x + B\cosh x = -Ae^{-x}$ **B1**

IF $B < -A$, then $A\sinh x + B\cosh x$ can be written $R\cosh(x+\gamma)$-

requiring $A = R\sinh\gamma$ and $B = R\cosh\gamma$, so $R = -\sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$ **B1 (5)**

(i) $y = a\tanh x + b = \operatorname{sech} x$ **M1**

Thus $a\sinh x + b\cosh x = 1$ **A1**

So $\sqrt{b^2 - a^2}\cosh\left(x + \tanh^{-1}\frac{a}{b}\right) = 1$ using first result of question M1

$$\cosh\left(x + \tanh^{-1}\frac{a}{b}\right) = \frac{1}{\sqrt{b^2 - a^2}}$$

$$x + \tanh^{-1}\frac{a}{b} = \pm\cosh^{-1}\left(\frac{1}{\sqrt{b^2 - a^2}}\right)$$

M1

and so

$$x = \pm\cosh^{-1}\left(\frac{1}{\sqrt{b^2 - a^2}}\right) - \tanh^{-1}\frac{a}{b}$$

A1* (5)

(ii)

$$x = \sinh^{-1}\left(\frac{1}{\sqrt{a^2 - b^2}}\right) - \tanh^{-1}\frac{b}{a}$$

M1 A1 (2)

(iii) For intersection to occur at two distinct points, we require two solutions to exist to the equations considered simultaneously. Considering the two graphs, there can be at most only one intersection, which would occur for $x > 0$, if $b \leq 0$.

Thus we require $b > a$ and $\left(\frac{1}{\sqrt{b^2-a^2}}\right) > 1$ **M1**

That is $a < b < \sqrt{a^2+1}$. **A1**

Similarly vice versa, if these conditions apply, then there are two solutions and hence two intersections. **E1 (3)**

(iv) To touch, we require two coincident solutions. i.e. $\left(\frac{1}{\sqrt{b^2-a^2}}\right) = 1$

That is $b = \sqrt{a^2+1}$, and equally, if this applies then they will touch, **E1**

so

$$x = -\tanh^{-1}\frac{a}{\sqrt{a^2+1}}$$

M1

and thus $y = a\tanh\left(-\tanh^{-1}\frac{a}{\sqrt{a^2+1}}\right) + \sqrt{a^2+1} = -\frac{a^2}{\sqrt{a^2+1}} + \sqrt{a^2+1} = \frac{1}{\sqrt{a^2+1}}$

A1 **M1** **A1 (5)**