STEP III, 2016 , Q5 MS

5. The binomial expansion, evaluated for $x = 1$, appreciating that terms are symmetrical contains two terms equal to the LHS of the inequality, and so truncating to them gives double the required result in (i). Appreciating that $\frac{(2m+1)!}{(m+1)!m!}$ is an integer and that if $m + 1 < p \leq 2m + 1$, with p a prime, implies p divides the numerator and not the denominator of this expression and hence divides the integer then can be extended for all such primes yielding the result, with the deduction following from (i). For (iii), it can be shown that $m + 1 \leq 2m$ and writing $P_{1,2m+1}$ as

$P_{1,m+1}P_{m+1,2m+1}$, combining the given result and (ii), the desired result is obtained. Part (iv) is obtained by use of strong induction with the supposition, $P_{1,m} < 4^m$ for all $m \leq k$ for some particular $k \geq 2$, and considering the cases k even and odd separately and making use of (iii).

5. (i)

$$(1+x)^{2m+1} = 1 + \binom{2m+1}{1}x + \cdots + \binom{2m+1}{m}x^m + \binom{2m+1}{m+1}x^{m+1} + \cdots + x^{2m+1}$$

B1

$$= 1 + \binom{2m+1}{1}x + \cdots + \binom{2m+1}{m}x^m + \binom{2m+1}{m}x^{m+1} + \cdots + x^{2m+1}$$

M1

$$x = 1 \Rightarrow 2^{2m+1} = 2\left[1 + \binom{2m+1}{1} + \cdots + \binom{2m+1}{m}\right] > 2\binom{2m+1}{m}$$

M1

and hence $\binom{2m+1}{m} < 2^{2m}$ **A1* (4)**

(ii) $\binom{2m+1}{m} = \frac{(2m+1)!}{(m+1)!m!}$ is an integer. **E1**

If p is a prime greater than $m+1$ and less than or equal to $2m+1$, then p is a factor of $(2m+1)!$

E1

and is not a factor of $(m+1)!\,m!$, **E1** and so it is a factor of $\binom{2m+1}{m}$. **E1**

Therefore, $P_{m+1,2m+1}$, which is the product of such primes, divides $\binom{2m+1}{m}$. **E1**

Hence, $kP_{m+1,2m+1} = \binom{2m+1}{m}$ where $k \geq 1$ is an integer, **M1** and hence

$P_{m+1,2m+1} = \frac{1}{k}\binom{2m+1}{m} < \frac{1}{k}2^{2m}$, i.e. $P_{m+1,2m+1} < 2^{2m}$ **A1* (7)**

(iii) $P_{1,2m+1} = P_{1,m+1}P_{m+1,2m+1}$ **M1**

$m \geq 1 \Rightarrow m + m \geq m+1$ i.e. $m+1 \leq 2m$ and so $P_{1,m+1} < 4^{m+1}$ applying given condition **E1**

By (ii), $P_{m+1,2m+1} < 2^{2m} = 4^m$ **M1**

Thus, $P_{1,2m+1} < 4^{m+1}4^m = 4^{2m+1}$ as required. **A1* (4)**

(iv) Suppose $P_{1,m} < 4^m$ for all $m \leq k$ for some particular $k \geq 2$. **E1**

Then if $k = 2m$, $P_{1,k+1} < 4^{k+1}$ by (iii). **E1**

$P_{1,2m+2} = P_{1,2m+1} < 4^{2m+1} < 4^{2m+2}$ (equality as $2m+2$ is not prime) using (iii). **E1**

So if $k = 2m+1$, $P_{1,k+1} < 4^{k+1}$. **E1**

$P_{1,2} = 2 < 4^2$ and hence required result is true by principle of mathematical induction. **dE1 (5)**