STEP III, 2016 , Q5

5 **(i)** By considering the binomial expansion of $(1+x)^{2m+1}$, prove that

$$\binom{2m+1}{m} < 2^{2m},$$

for any positive integer m.

(ii) For any positive integers r and s with $r < s$, $P_{r,s}$ is defined as follows: $P_{r,s}$ is the product of all the prime numbers greater than r and less than or equal to s, if there are any such primes numbers; if there are no such primes numbers, then $P_{r,s} = 1$.

For example, $P_{3,7} = 35$, $P_{7,10} = 1$ and $P_{14,18} = 17$.

Show that, for any positive integer m, $P_{m+1\,,2m+1}$ divides $\binom{2m+1}{m}$, and deduce that

$$P_{m+1\,,2m+1} < 2^{2m}.$$

(iii) Show that, if $P_{1,k} < 4^k$ for $k = 2, 3, \ldots, 2m$, then $P_{1,2m+1} < 4^{2m+1}$.

(iv) Prove that $\mathrm{P}_{1,n} < 4^n$ for $n \geqslant 2$.