STEP III, 2016 , Q4 MS

4. The considered expression equates to $\frac{(x-1)x^r}{(1+x^r)(1+x^{r+1})}$ and so, by the method of differences,

$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{1}{(x-1)}\left[\frac{1}{1+x} - \frac{1}{1+x^{N+1}}\right]$, and letting $N \to \infty$, the desired result is obtained.
Writing $\operatorname{sech}(ry)$ as $\frac{2e^{-ry}}{1+e^{-2ry}}$ and similarly $\operatorname{sech}\big((r+1)y\big)$, the result of part (i) with $x = e^{-2y}$ can be used to obtain the result. Care needs to be taken to write $\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big)$ as

$2\left[\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}\big((r+1)y\big) + \operatorname{sech} y\right]$ which with the previous deduction of (ii) can be simplified to $2 \operatorname{cosech} y$.

4. (i)

$$\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}} = \frac{(1+x^{r+1}) - (1+x^r)}{(1+x^r)(1+x^{r+1})} = \frac{(x-1)x^r}{(1+x^r)(1+x^{r+1})}$$

B1

Therefore

$$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{1}{(x-1)}\sum_{r=1}^{N}\left(\frac{1}{1+x^r} - \frac{1}{1+x^{r+1}}\right) = \frac{1}{(x-1)}\left[\frac{1}{1+x} - \frac{1}{1+x^{N+1}}\right]$$

M1 **M1 A1**

As $N \to \infty$, as $|x| < 1$, $\frac{1}{1+x^{N+1}} \to 1$ **M1**

So

$$\sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{1}{(x-1)}\left[\frac{1}{1+x} - 1\right] = \frac{1}{(x-1)}\left[\frac{1-1-x}{1+x}\right] = \frac{-x}{x^2-1} = \frac{x}{1-x^2}$$

A1 * (6)

(ii)

$$\operatorname{sech}(ry) = \frac{1}{\cosh(ry)} = \frac{2}{e^{ry} + e^{-ry}} = \frac{2e^{-ry}}{1 + e^{-2ry}}$$

$$\operatorname{sech}((r+1)y) = \frac{2e^{-(r+1)y}}{1 + e^{-2(r+1)y}}$$

M1

Thus

$$\operatorname{sech}(ry)\operatorname{sech}((r+1)y) = \frac{4e^{-y}e^{-2ry}}{(1 + e^{-2ry})(1 + e^{-2(r+1)y})}$$

A1

So if $x = e^{-2y}$, **M1**

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) = 4e^{-y}\sum_{r=1}^{\infty} \frac{x^r}{(1 + x^r)(1 + x^{r+1})} = 4e^{-y}\frac{x}{1 - x^2}$$

A1 **A1**

Thus

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) = 4e^{-y}\frac{e^{-2y}}{1 - e^{-4y}} = 2e^{-y}\frac{2}{e^{2y} - e^{-2y}}$$

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) = 2e^{-y}\operatorname{csch}(2y)$$

M1 A1* (7)

(iii)

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) = 2\left[\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \operatorname{sech} y\right]$$

M1 A1

$$= 2[2e^{-y}\operatorname{csch}(2y) + \operatorname{sech} y] = 2\left[\frac{2e^{-y}}{\sinh 2y} + \frac{1}{\cosh y}\right] = 2\left[\frac{e^{-y}}{\sinh y \cosh y} + \frac{1}{\cosh y}\right]$$

A1 **M1 A1**

$$= \frac{2}{\cosh y}\left[\frac{2e^{-y} + e^{y} - e^{-y}}{2\sinh y}\right] = \frac{2}{\cosh y}\left[\frac{2\cosh y}{2\sinh y}\right] = 2\operatorname{csch} y$$

M1 **A1 (7)**