STEP III, 2016 , Q3 MS

3. Differentiating, multiplying by denominators and dividing by the exponential function, gives $[Q(P+P')-PQ'](x+1)^2=(x^3-2)Q^2$ which, invoking the factor theorem, gives the first required result. Denoting the degree of P by p and that of Q by q in this expression yields $p+q+2=2q+3$ and hence the desired result in (i). Furthermore, in the given case, substitution in the same result and postulating $P(x)=ax^2+bx+c$ yields consistent equations for a, b and c and thus $P(x)=x^2-2x$.
For part (ii), commencing as in part (i) demonstrates again that Q has a factor $(x+1)$ as $[Q(P+P')-PQ'](x+1)=Q^2$. Supposing $Q(x)=(x+1)^nS(x)$, where $n\geq 2$ and $S(-1)\neq 0$, with $P(-1)\neq 0$ and substituting in the expression already derived leads to a contradiction.

3. (i)

$$\frac{d}{dx}\left(\frac{Pe^x}{Q}\right)=\frac{Q(Pe^x+P'e^x)-Pe^xQ'}{Q^2}$$

It is required that $\frac{d}{dx}\left(\frac{Pe^x}{Q}\right)=\frac{x^3-2}{(x+1)^2}e^x$ so it follows that

$$[Q(Pe^x+P'e^x)-Pe^xQ'](x+1)^2=(x^3-2)e^xQ^2$$

M1 A1

Thus,

$$[Q(P+P')-PQ'](x+1)^2=(x^3-2)Q^2$$

Letting $x=-1$, $0=-3[Q(-1)]^2$ so $Q(-1)=0$ and thus Q has a factor $(x+1)$ as required.

M1 A1 (4)

Suppose that the degree of $P(x)$ is p and that of $Q(x)$ is q. **M1**

Then the degree of $P'(x)$ is $p-1$ and of $Q'(x)$ is $q-1$
So $P+P'$ has degree p, $Q(P+P')$ has degree $p+q$, PQ' has degree $p+q-1$,
$[Q(P+P')-PQ']$ has degree $p+q$, and thus $[Q(P+P')-PQ'](x+1)^2$ has degree $p+q+2$

A1

$(x^3-2)Q^2$ has degree $2q+3$

Thus $p+q+2=2q+3$ which means that $p=q+1$ as required. **A1 (3)**

If $Q(x) = x + 1$, $Q'(x) = 1$, and so

$$(x + 1)(P + P') - P = (x^3 - 2)$$

M1 A1

That is $xP + (x + 1)P' = (x^3 - 2)$

$P(x) = ax^2 + bx + c$ and so $P'(x) = 2ax + b$ **B1**

Therefore $x(ax^2 + bx + c) + (x + 1)(2ax + b) = (x^3 - 2)$ **M1**

and equating coefficients

$$a = 1\,, b + 2a = 0, c + 2a + b = 0, b = -2$$

A1

These equations are consistent, with $a = 1, b = -2, c = 0$ so $P(x) = x^2 - 2x$ **A1 (6)**

(ii) For such *P* and *Q* to exist, $\frac{d}{dx}\left(\frac{Pe^x}{Q}\right) = \frac{1}{x+1}e^x$

and so

$$[Q(Pe^x + P'e^x) - Pe^xQ'](x + 1) = e^xQ^2$$

M1

and

$$[Q(P + P') - PQ'](x + 1) = Q^2$$

A1

Letting $x = -1\,, 0 = [Q(-1)]^2$ so $Q(-1) = 0$ and thus *Q* has a factor $(x + 1)$ as before in (i).

However, letting $Q(x) = (x + 1)R(x)$, then **M1**

$$[(x + 1)R(P + P') - P(R + (x + 1)R')](x + 1) = (x + 1)^2R^2$$

and so

$$(x + 1)(RP + RP' - PR') - PR = (x + 1)R^2$$

Letting $x = -1$, $P(-1)R(-1) = 0$, but $P(-1) \neq 0$ as *P* and *Q* have no common factors, and so

$R(-1) = 0$ which means that *R* in turn has a factor $(x + 1)$. **A1**

Thus Q must have a factor of $(x + 1)^2$.

Suppose $Q(x) = (x + 1)^n S(x)$, where $n \geq 2$ and $S(-1) \neq 0$ **M1**

Then

$$[Q(P+P')-PQ'](x+1)=Q^2$$

becomes

$$(x+1)[(x+1)^n S(P+P')-P(n(x+1)^{n-1}S+(x+1)^n S')]=(x+1)^{2n}S^2$$

Dividing by the factor $(x+1)^n$ gives,

$$[(x+1)S(P+P')-P(nS+(x+1)S')]=(x+1)^n S^2$$

A1

Letting $x=-1$, $nP(-1)S(-1)=0$, but $n\neq 0$, $P(-1)\neq 0$ and $S(-1)\neq 0$ giving a contradiction and hence no such P and Q can exist. **E1**