# STEP III, 2016 , Q2 MS

2. There are numerous correct ways through this question. Working parametrically with $x = at^2$ , $y = 2at$ gives a normal as $tx + y = at^3 + 2at$ and imposing that this passes through $(ap^2, 2ap)$ yields $t^2 + tp + 2 = 0$ (*) which has roots $q$ and $r$, the former giving (i). As a consequence, $r + q = -p$ and $rq = 2$ , so that QR, $2x - (r+q)y + 2aqr = 0$ , simplifies to $2x + py + 4a = 0$ , and thus passes through $(-2a, 0)$ for (ii). T can be shown to be $\left(-a, \frac{-2a}{p}\right)$, which, of course, lies on $x = -a$ , and as (*) had two real distinct roots, $q$ and $r$, $p^2 - 8 > 0$ , which yields $\left|\frac{-2a}{p}\right| < \frac{a}{\sqrt{2}}$.

2. (i) $x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

$y = 2at \Rightarrow \frac{dy}{dt} = 2a$

So $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Thus, the gradient of the normal at $Q$ is $-q$ . **M1 A1**

But this normal is the chord $PQ$ and so has gradient $\frac{2ap-2aq}{ap^2-aq^2} = \frac{2a(p-q)}{a(p-q)(p+q)} = \frac{2}{p+q}$ **M1 A1**

So $-q = \frac{2}{p+q}$ which rearranges to $q^2 + qp + 2 = 0$ **A1* (5)**

(ii) Similarly, $r^2 + rp + 2 = 0$

**B1**

Making $p$ the subject of each result, $p = \frac{-(2+q^2)}{q} = \frac{-(2+r^2)}{r}$

So $2r - 2q + q^2r - qr^2 = 0$

$2(r-q) - qr(r-q) = 0$

As $(r-q) \neq 0$ , $qr = 2$ **M1 A1**

The line $QR$ is $\frac{y-2aq}{x-aq^2} = \frac{y-2ar}{x-ar^2}$

That is $xy - 2aqx - ar^2y + 2a^2qr^2 = xy - 2arx - aq^2y + 2a^2q^2r$

$2ax(r-q) - a(r^2-q^2)y + 2a^2qr(r-q) = 0$

Again, as $(r-q) \neq 0$ , and $\neq 0$ , $2x - (r+q)y + 2aqr = 0$ **M1 A1**

Because $qr = 2$ , $QR$ is $2x - (r+q)y + 4a = 0$

So when $y = 0$ , $x = -2a$ and thus $(-2a, 0)$ is a suitable fixed point. **B1 (6)**

(iii) Because $q^2 + qp + 2 = 0$ and $r^2 + rp + 2 = 0$ subtracting gives

$q^2 - r^2 + qp - rp = 0$

Again, as $(r-q) \neq 0$ , $q + r + p = 0$ **M1 A1**

So the line *QR* is $2x + py + 4a = 0$ **M1**

The line *OP* is $y = \frac{2ap}{ap^2}x$ i.e. $y = \frac{2}{p}x$ **B1**

Thus the intersection of QR and OP is at $\left(-a, \frac{-2a}{p}\right)$, which lies on $x = -a$. **B1 (5)**

$\frac{-2}{p} = \frac{2q}{2+q^2} = \frac{2r}{2+r^2}$

Suppose $k = \frac{-2}{p}$, then $kq^2 - 2q + 2k = 0$ **M1**

As this equation has two distinct real roots, *q* and *r*, then the discriminant is positive **M1**

and so

$4 - 8k^2 > 0$ **A1** so $k^2 < \frac{1}{2}$ that is $-\frac{1}{\sqrt{2}} < k < \frac{1}{\sqrt{2}}$ which means that the distance of that point of intersection is less than $\frac{a}{\sqrt{2}}$ from the x axis. **A1* (4)**