# STEP III, 2016 , Q1 MS

1. Part (i) is most simply dealt with by the suggested method, change of variable, and it is worth completing the square in the denominator to simplify the algebra leading to a trivial integral. Part (ii) can either be attempted immediately using integration by parts, starting from $I_n$ and obtaining $\int_{-\infty}^{\infty} \frac{2nx(x+a)}{(x^2+2ax+b)^{n+1}}\, dx$ and then writing the numerator as $2n(x^2+2ax+b) - na(2x+2a) - 2n(b-a^2)$. Alternatively, use of the same substitution in $I_{n+1}$ as in part (i) leads to the need to integrate $\cos^{2n} u$, which in turn can be written as $\cos^{2n-2} u(1-\sin^2 u) = \cos^{2n-2} u - \cos^{2n-2} u \sin u . \sin u$, with the second term being susceptible to integration by parts. Part (iii) follows from the previous parts by induction using part (ii) to achieve the inductive step and (i) the base case.

1. (i)

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{x^2+2ax+b}\,dx$$

$$x + a = \sqrt{b-a^2}\tan u$$

$$\frac{dx}{du} = \sqrt{b-a^2}\sec^2 u$$

**M1 A1**

$$I_1 = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u}{(b-a^2)\tan^2 u + (b-a^2)}\,du = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u}{(b-a^2)\sec^2 u}\,du$$

**M1 A1** **M1**

$$I_1 = \frac{1}{\sqrt{b-a^2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 1\,du = \frac{\pi}{\sqrt{b-a^2}}$$

**A1 *** **(6)**

(ii)

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2+2ax+b)^n}\,dx = \left[\frac{x}{(x^2+2ax+b)^n}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-2nx(x+a)}{(x^2+2ax+b)^{n+1}}\,dx$$

**M1 A1**

$$I_n = 2n\int_{-\infty}^{\infty} \frac{x^2+ax}{(x^2+2ax+b)^{n+1}}\,dx = 2n\int_{-\infty}^{\infty} \frac{x^2+2ax+b}{(x^2+2ax+b)^{n+1}} - \frac{ax+b}{(x^2+2ax+b)^{n+1}}\,dx$$

**M1 A1**

$$I_n = 2nI_n - 2n\int_{-\infty}^{\infty} \frac{\frac{a}{2}(2x+2a)}{(x^2+2ax+b)^{n+1}} + \frac{(b-a^2)}{(x^2+2ax+b)^{n+1}}\,dx$$

**M1**

$$I_n = 2nI_n - 2n\left[\frac{\frac{-a}{2n}}{(x^2+2ax+b)^n}\right]_{-\infty}^{\infty} - 2n(b-a^2)I_{n+1}$$

**A1**

$$2n(b-a^2)I_{n+1} = (2n-1)I_n$$

**A1 *** **(7)**

(iii) Suppose $I_k = \frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}}\binom{2k-2}{k-1}$ for some integer k, $k \geq 1$

**B1**

Then $I_{k+1} = \frac{2k-1}{2k(b-a^2)}\frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}}\binom{2k-2}{k-1} = \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \times \frac{2(2k-1)}{k}\binom{2k-2}{k-1}$

**M1**

$$\frac{2(2k-1)}{k}\binom{2k-2}{k-1} = \frac{2(2k-1)}{k}\frac{(2k-2)!}{(k-1)!\,(k-1)!} = \frac{2k(2k-1)}{kk}\frac{(2k-2)!}{(k-1)!\,(k-1)!} = \binom{2k}{k}$$

so result true for k+1. **M1 A1**

For $n = 1$,

$$\frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}}\binom{2n-2}{n-1} = \frac{\pi}{(b-a^2)^{\frac{1}{2}}}\binom{0}{0} = \frac{\pi}{(b-a^2)^{\frac{1}{2}}}$$

**M1 A1**

which is the correct result.

So result has been proved by (principle of)(mathematical) induction. **dB1 (7)**