STEP III, 2016 , Q13 MS

13. Showing that $X - a$ has the same kurtosis as X requires the expectations of $X - a$, $(X - a - \mu + a)^2$, and $(X - a - \mu + a)^4$ to be obtained and substituted. For part (i), the numerator can be obtained by an integration by parts reducing the integral to the one that gives the variance. Expanding T^4 as $\sum(Y_r{}^4 + 4Y_r{}^3Y_s + 6Y_r{}^2Y_s{}^2 + 12Y_sY_tY_r{}^2 + 24Y_rY_sY_tY_u)$, where the summation is over all values without repetition, and taking the expectation of these terms gives the requested result in part (ii). Defining $Y_i = X_i - \mu$, the kurtosis of Y_i by the first result gives $E(Y_i{}^4) = (3 + \kappa)\sigma^4$ and defining T as in (ii), the kurtosis of $\sum_{i=1}^{n} X_i$ is, using the result of (ii), $\frac{n(3+\kappa)\sigma^4 + 3n(n-1)\sigma^2\sigma^2}{n^2\sigma^4} - 3$ giving the required answer.

13. Let $Y = X - a$, then $\mu_Y = E(Y) = E(X-a) = E(X) - a = \mu - a$ **B1**

$$E((Y-\mu_Y)^4) = E((X-a-\mu+a)^4) = E((X-\mu)^4)$$

B1

$$\sigma_Y{}^2 = E((Y-\mu_Y)^2) = E((X-a-\mu+a)^2) = E((X-\mu)^2) = \sigma^2$$

B1

so the kurtosis of $X - a$ is

$$\frac{E((Y-\mu_Y)^4)}{\sigma_Y{}^4} - 3 = \frac{E((X-\mu)^4)}{\sigma^4} - 3$$

which is the same as that for X **B1* (4)**

(i) If $X \sim N(0, \sigma^2)$ then it has pdf

$$\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2}$$

So

$$E((X-\mu)^4) = \int_{-\infty}^{\infty} x^4 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} x^3 x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

M1 A1

By parts,

$$\int_{-\infty}^{\infty} x^3 x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx = \left[x^3 \times -\sigma^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 3x^2 \times -\sigma^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

M1 A1

$$= 0 + 3\sigma^2\sigma^2 = 3\sigma^4$$

A1

So the kurtosis is

$$\frac{3\sigma^4}{\sigma^4} - 3 = 0$$

as required. **(5)**

(ii)

$$T^4 = \left(\sum_{r=1}^{n} Y_r\right)^4 = \sum\left(Y_r{}^4 + 4Y_r{}^3Y_s + 6Y_r{}^2Y_s{}^2 + 12Y_sY_tY_r{}^2 + 24Y_rY_sY_tY_u\right)$$

M1 A1

where the summation is over all values without repetition.
As the Ys are independent, the expectation of products are products of expectations and as

$E(Y) = 0$,

$$E(T^4) = E\left(\sum\left(Y_r{}^4 + 6Y_r{}^2Y_s{}^2\right)\right) = E\left(\sum_{r=1}^{n} Y_r{}^4\right) + E\left(\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} 6Y_r{}^2Y_s{}^2\right)$$

M1

$$= \sum_{r=1}^{n} E\left(Y_r{}^4\right) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} E\left(Y_r{}^2\right)E\left(Y_s{}^2\right)$$

A1* (4)

(iii)

$$\frac{E((X_i-\mu)^4)}{\sigma^4} - 3 = \kappa$$

Let $Y_i = X_i - \mu$ then by the first result, the kurtosis of Y_i is ,

i.e.

$$\frac{E\left(Y_i{}^4\right)}{\sigma^4} - 3 = \kappa$$

so $E\left(Y_i{}^4\right) = (3+\kappa)\sigma^4$ **M1 A1**

$$E\left(\sum_{i=1}^{n} X_i\right) = n\mu$$

and

$$Var\left(\sum_{i=1}^{n} X_i\right) = n\sigma^2$$

B1

so the kurtosis of

$$\sum_{i=1}^{n} X_i$$

is

$$\frac{E\left(\left(\sum_{i=1}^{n} X_i - n\mu\right)^4\right)}{(n\sigma^2)^2} - 3 = \frac{E\left(\left(\sum_{i=1}^{n} Y_i\right)^4\right)}{n^2\sigma^4} - 3$$

M1

Let

$$T = \sum_{r=1}^{n} Y_r$$

Then we require

$$\frac{E(T^4)}{n^2\sigma^4} - 3$$

which by (ii) is

$$\frac{\sum_{r=1}^{n} E\left({Y_r}^4\right) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} E\left({Y_r}^2\right)E\left({Y_s}^2\right)}{n^2\sigma^4} - 3$$

M1 A1

$$= \frac{n(3+\kappa)\sigma^4 + 3n(n-1)\sigma^2\sigma^2}{n^2\sigma^4} - 3 = \frac{3+\kappa+3(n-1)-3n}{n} = \frac{\kappa}{n}$$

A1* (7)