STEP III, 2016 , Q13

13 Given a random variable X with mean μ and standard deviation σ, we define the *kurtosis*, κ, of X by

$$\kappa = \frac{\mathrm{E}\big((X-\mu)^4\big)}{\sigma^4} - 3\,.$$

Show that the random variable $X - a$, where a is a constant, has the same kurtosis as X.

(i) Show by integration that a random variable which is Normally distributed with mean 0 has kurtosis 0.

(ii) Let $Y_1, Y_2, \ldots, Y_n$ be n independent, identically distributed, random variables with mean 0, and let $T = \sum_{r=1}^{n} Y_r$. Show that

$$\mathrm{E}(T^4) = \sum_{r=1}^{n} \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mathrm{E}(Y_s^2)\mathrm{E}(Y_r^2)\,.$$

(iii) Let $X_1, X_2, \ldots, X_n$ be n independent, identically distributed, random variables each with kurtosis κ. Show that the kurtosis of their sum is $\dfrac{\kappa}{n}$.