STEP III, 2016 , Q12

12 Let X be a random variable with mean μ and standard deviation σ. *Chebyshev's inequality*, which you may use without proof, is

$$\mathrm{P}\left(|X-\mu|>k\sigma\right)\leqslant\frac{1}{k^2}\,,$$

where k is any positive number.

(i) The probability of a biased coin landing heads up is 0.2. It is thrown $100n$ times, where n is an integer greater than 1. Let α be the probability that the coin lands heads up N times, where $16n\leqslant N\leqslant 24n$.

Use Chebyshev's inequality to show that

$$\alpha\geqslant 1-\frac{1}{n}\,.$$

(ii) Use Chebyshev's inequality to show that

$$1+n+\frac{n^2}{2!}+\cdots+\frac{n^{2n}}{(2n)!}\geqslant\left(1-\frac{1}{n}\right)\mathrm{e}^n\,.$$