STEP III, 2016 , Q1

1 Let

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2ax + b)^n} \, \mathrm{d}x\,,$$

where a and b are constants with $b > a^2$, and n is a positive integer.

(i) By using the substitution $x + a = \sqrt{b - a^2}\,\tan u$, or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b - a^2}}\,.$$

(ii) Show that $2n(b - a^2)\, I_{n+1} = (2n - 1)\, I_n\,.$

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b - a^2)^{n - \frac{1}{2}}} \binom{2n-2}{n-1}.$$