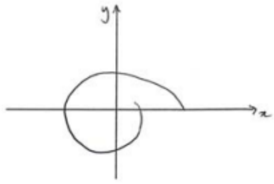
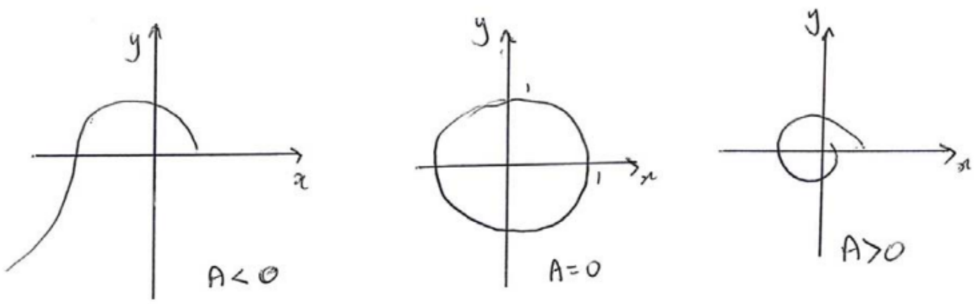
STEP III, 2015 , Q8 MS

8. Transforming the differential equation in part (i) is made by substituting for x and y as given, for $\frac{dy}{dx}$ using $\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$ and a similar result for $\frac{dx}{d\theta}$, and then simplifying the algebra by multiplying out and collecting like terms bearing in mind that a factor r can be cancelled as $r \neq 0$. The transformed equation can be solved by separating variables or using an integrating factor, to give $r = ke^{-\theta}$, the sketch of which is



The same techniques for part (ii) yields a differential equation $r - r^3 + \frac{dr}{d\theta} = 0$ which is solved by separating the variables and then employing partial fractions giving a variety of possible solution sketches



($A \neq -1$ but it is possible to consider $A < -1$ in which case $\theta < 0$)

8. (i) $x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$ **M1A1**

and $y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$ **M1A1**

Thus $(y+x)\frac{dy}{dx} = y - x$ becomes $(r\sin\theta + r\cos\theta)\frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} = r\sin\theta - r\cos\theta$ **M1**

That is $(\sin\theta + \cos\theta)\left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right) = (\sin\theta - \cos\theta)\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)$

as $r > 0,\ r \neq 0$

Multiplying out and collecting like terms gives

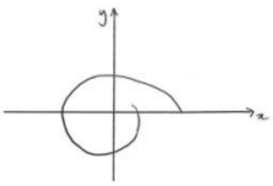
$$r(\cos^2\theta + \sin^2\theta) + \frac{dr}{d\theta}(\sin^2\theta + \cos^2\theta) = 0$$

which is $r + \frac{dr}{d\theta} = 0$. **M1A1* (7)**

So $re^{\theta} + \frac{dr}{d\theta}e^{\theta} = 0$ **M1**

and thus $re^{\theta} = k$, **A1**

$r = ke^{-\theta}$ **A1**



G1 (4)

(or alternatively $\int \frac{1}{r} dr = \int -d\theta$ **M1** so $\ln|r| = -\theta + c$ **A1** and hence $r = ke^{-\theta}$ **A1**)

(ii) $(y + x - x(x^2+y^2))\frac{dy}{dx} = y - x - y(x^2+y^2)$

becomes $(r\sin\theta + r\cos\theta - r^3\cos\theta)\frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} = r\sin\theta - r\cos\theta - r^3\sin\theta$

that is

$(\sin\theta + \cos\theta - r^2\cos\theta)\left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right) = (\sin\theta - \cos\theta - r^2\sin\theta)\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)$

Multiplying out and collecting like terms gives

$r(\cos^2\theta + \sin^2\theta - r^2(\cos^2\theta + \sin^2\theta)) + \frac{dr}{d\theta}(\sin^2\theta + \cos^2\theta) = 0$ **M1**

which is $r - r^3 + \frac{dr}{d\theta} = 0$. **A1**

$\int \frac{1}{r^3 - r} dr = \int d\theta$

$\int \frac{1}{r^3-r} dr = \int \frac{1}{r(r^2-1)} dr = \int \frac{1}{r(r-1)(r+1)} dr = \int d\theta$ **M1**

So $\int d\theta = \int \frac{1/2}{r-1} + \frac{-1}{r} + \frac{1/2}{r+1} dr$ **A1**

$\theta + k = \frac{1}{2}\ln\left|\frac{(r-1)(r+1)}{r^2}\right|$ **A1**

So

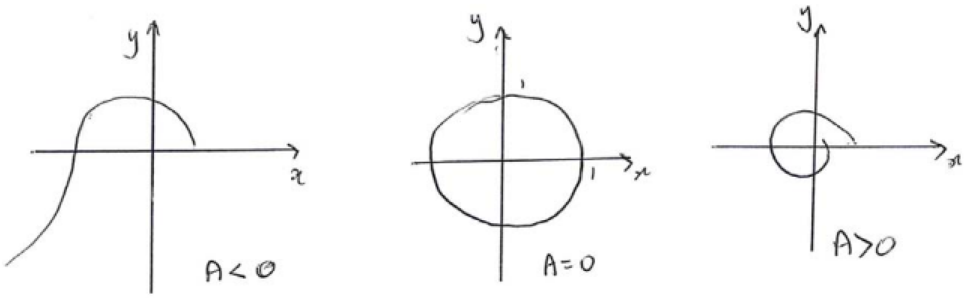
$$\left|\frac{r^2-1}{r^2}\right| = Ce^{2\theta}$$

with $C > 0$

$$r^2 = \frac{1}{1 \mp Ce^{2\theta}}$$

that is

$r^2 = \frac{1}{1+Ae^{2\theta}}$ **A1***



G1 G1 G1 (9)