STEP III, 2015 , Q7 MS

7. The opening result is simply achieved by following the given explanation for $D^2 f(x)$ with $f(x) = x^a$. Parts (i) and (ii) can both be shown usual the principle of mathematical induction with initial statements

"Suppose $D^k P(x)$ is a polynomial of degree r i.e. $D^k P(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0$ for some integer k ." and "Suppose $D^k (1-x)^m$ is divisible by $(1-x)^{m-k}$ i.e. $D^k (1-x)^m = f(x)(1-x)^{m-k}$ for some integer k , with $k < m-1$." Part (iii) is obtained by expressing $(1-x)^m$ in sigma notation (by the binomial theorem), then carrying out $D^n (1-x)^m$ using the idea in the stem, and finally invoking the result of part (ii) and then substituting $x = 1$.

7. $D^2 x^a = D(D(x^a)) = D\left(x \frac{d}{dx}(x^a)\right) = D(xax^{a-1})$ **M1**

$= D(ax^a) = x \frac{d}{dx}(ax^a) = xa^2 x^{a-1} = a^2 x^a$ **M1A1 (3)**

(i) Suppose $D^k P(x)$ is a polynomial of degree r i.e. $D^k P(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0$ for some integer k . **B1**

Then $D^{k+1} P(x) = D(a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0) = x \frac{d}{dx}(a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0)$

$= x(r a_r x^{r-1} + (r-1) a_{r-1} x^{r-2} + \cdots + a_1) = r a_r x^r + (r-1) a_{r-1} x^{r-1} + \cdots + a_1 x$

which is a polynomial of degree r . **M1A1**

Suppose $P(x) = b_r x^r + b_{r-1} x^{r-1} + \cdots + b_0$, then

$DP(x) = x \frac{d}{dx}(b_r x^r + b_{r-1} x^{r-1} + \cdots + b_0) = r b_r x^r + (r-1) b_{r-1} x^{r-1} + \cdots + b_1 x$ so the result is true for $n = 1$, **M1A1**

and we have shown that if it is true for $n = k$, it is true for $n = k+1$. Hence by induction, it is true for any positive integer . **B1 (6)**

(ii) Suppose $D^k(1-x)^m$ is divisible by $(1-x)^{m-k}$ i.e. $D^k(1-x)^m = f(x)(1-x)^{m-k}$ for some integer k , with $k < m-1$. **B1**

Then $D^{k+1}(1-x)^m = D\big(f(x)(1-x)^{m-k}\big) = x\frac{d}{dx}\big(f(x)(1-x)^{m-k}\big)$

$= x\big(f'^{(x)}(1-x)^{m-k} - (m-k)f(x)(1-x)^{m-k-1}\big)$

$= x(1-x)^{m-k-1}\left(f'^{(x)}(1-x) - (m-k)f(x)\right)$ which is divisible by $(1-x)^{m-(k+1)}$. **M1A1**

$D(1-x)^m = x\frac{d}{dx}((1-x)^m) = -mx(1-x)^{m-1}$ so result is true for $n = 1$. **M1A1**

We have shown that if it is true for $n = k$, it is true for $n = k+1$. Hence by induction, it is true for any positive integer $< m$. **B1 (6)**

(iii)

$(1-x)^m = \sum_{r=0}^{m}\binom{m}{r}(-x)^r = \sum_{r=0}^{m}(-1)^r\binom{m}{r}x^r$ **M1**

So

$D^n(1-x)^m = \sum_{r=0}^{m}(-1)^r\binom{m}{r}D^n x^r = \sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n x^r$ **M1A1**

But by (ii), $D^n(1-x)^m$ is divisible by $(1-x)^{m-n}$ and so $D^n(1-x)^m = g(x)(1-x)^{m-n}$, and thus if $x = 1$, $D^n(1-x)^m = 0$, and hence

$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n = 0$ **M1A1* (5)**