STEP III, 2015 , Q6 MS

6. Treating the equations for u and v as simultaneous equations for w and z, one finds that $w = \frac{u \pm \sqrt{2v - u^2}}{2}$ and $z = \frac{u \mp \sqrt{2v - u^2}}{2}$ which demonstrates that if $u \in \mathbb{R}$ and $u^2 \leq 2v$, i.e. $v \in \mathbb{R}$, w and z are real. If w and z are real, then u and v are (trivially) and $2v - u^2 = (w - z)^2 \geq 0$.

In (ii), the first two equations yield $3wz = 1$, making it possible to write the third equation as $u(u^2 - 1) = \lambda(u - 1)$ which has an obvious factor of $(u - 1)$ leading to $u = 1$ or $u = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2}$ from the quadratic equation. If one of the solutions of the quadratic equation gives the same root $u = 1$, then there are not three possible values, i.e. if $\lambda = 2$. From the first part of the question, for w and z to be real, we would want u to be real, $u^2 - \frac{2}{3}$ to be real, and $u^2 \leq 2\left(u^2 - \frac{2}{3}\right)$, in other words $u^2 \geq \frac{4}{3}$. So a counter-example could be $u = 1$ giving $2w^2 - 2w + \frac{2}{3} = 0$ which has a negative discriminant.

6. (i) $w, z \in \mathbb{R} \Longrightarrow u, v \in \mathbb{R}$ **B1**

For $w, z \in \mathbb{R}$, we require to solve $w + z = u$, $w^2 + z^2 = v$ **M1**

$$w^2 + (u - w)^2 = v$$

$$2w^2 - 2uw + (u^2 - v) = 0$$

$$w = \frac{2u \pm \sqrt{4u^2 - 8u^2 + 8v}}{4} = \frac{u \pm \sqrt{2v - u^2}}{2}$$

$$z = \frac{u \mp \sqrt{2v - u^2}}{2}$$

M1A1

So for, $z \in \mathbb{R}$, as $u = w + z$ must be real, $v = w^2 + z^2$ must be real, and $2v - u^2 \geq 0$

i.e. $u^2 \leq 2v$ **B1* (5)**

(ii) $u = w + z \implies u^2 = w^2 + z^2 + 2wz$ so if $w^2 + z^2 - u^2 = -\frac{2}{3}$, then $-2wz = -\frac{2}{3}$

so $3wz = 1$ **M1A1**

$$w^3 + z^3 = (w + z)(w^2 + z^2 - wz) = u(u^2 - 3wz) = u(u^2 - 1)$$

M1A1

Thus if $w^3 + z^3 - \lambda u = -\lambda$, $u(u^2 - 1) = \lambda(u - 1)$ **M1A1**

Thus $(u - 1)(u(u + 1) - \lambda) = 0$, **M1**

$(u - 1)(u^2 + u - \lambda) = 0$ **M1A1**

Thus $u = 1$ or $u = \frac{-1 \pm \sqrt{1+4\lambda}}{2}$

So as $\lambda \in \mathbb{R}$ and $\lambda > 0$, the values of u are real. **B1**

There are three distinct values of u unless $\frac{-1 \pm \sqrt{1+4\lambda}}{2} = 1$ in which case $\pm\sqrt{1 + 4\lambda} = 3$, i.e. $\lambda = 2$

M1A1 (12)

For $w, z \in \mathbb{R}$, from (i) we require $u \in \mathbb{R}$ which it is, $u^2 - \frac{2}{3} \in \mathbb{R}$ which it is, and $u^2 \leq 2\left(u^2 - \frac{2}{3}\right)$ in other words $u^2 \geq \frac{4}{3}$. **M1**

So w and z need not be real. A counterexample would be $u = 1$ **B1**

for then $w + z = 1$, $w^2 + z^2 = \frac{1}{3}$, so $w^2 + (1 - w)^2 = \frac{1}{3}$, i.e. $2w^2 - 2w + \frac{2}{3} = 0$ in which case the discriminant is $-\frac{4}{3} < 0$ so $w \notin \mathbb{R}$. **B1 (3)**