STEP III, 2015 , Q5 MS

5. (i) Step 3 is straightforward on the basis of steps 1 and 2, noting that no lowest terms restriction need be made in part 1. Step 5 requires that the given expression is a positive integer as well as well as being integer when multiplied by root two. Step 6 requires justification that $\sqrt{2}-1<1$.

(ii) The rationality of $2^{2/3}$ on the basis of $2^{1/3}$ being rational is simply obtained by squaring the latter, and the opposite implication can be made by squaring the former or dividing 2 by the former. To construct the similar argument, let the set T be the set of positive integers with the following property: n is in T if and only if $n2^{1/3}$ and $n2^{2/3}$ are integers, and taking t to be the smallest positive integer in that set, consider $t\left(2^{2/3}-1\right)$ to produce the argument.

5. (i) Having assumed that $\sqrt{2}$ is rational (step 1), $\sqrt{2}={}^{p}/_{q}$, where $p,q\in\mathbb{Z},q\neq 0$ **B1**

Thus from the definition of S (step 2), as $q\in\mathbb{Z}$ and $\sqrt{2}=q\times{}^{p}/_{q}=p\in\mathbb{Z}$, so $q\in S$ proving step 3. **B1 (2)**

If $k\in S$, then k is an integer and $k\sqrt{2}$ is an integer. **B1**

So $(\sqrt{2}-1)k=k\sqrt{2}-k$ is an integer, **B1**

and $(\sqrt{2}-1)k\sqrt{2}=2k-k\sqrt{2}$ which is an integer and so $(\sqrt{2}-1)k\in S$ proving step 5. **B1 (3)**

$1<\sqrt{2}<2$ and so **M1**

$0<\sqrt{2}-1<1$, and thus $0<(\sqrt{2}-1)k<k$ **A1**

and thus this contradicts step 4 that k is the smallest positive integer in S as $(\sqrt{2}-1)k$ has been shown to be a smaller positive integer and is in S. **A1 (3)**

(ii) If $2^{2/3}$ is rational, then $2^{2/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$

So $\left(2^{2/3}\right)^2 = \left({}^{p}/_{q}\right)^2$, that is $2^{4/3} = {}^{p^2}/_{q^2}$, which can be written $2 \times 2^{1/3} = {}^{p^2}/_{q^2}$ **M1**

and hence $2^{1/3} = {}^{p^2}/_{2q^2}$ proving that $2^{1/3}$ is rational. **A1**

If $2^{1/3}$ is rational, then $2^{1/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$ **M1**

and so $2^{2/3} = {}^{p^2}/_{q^2}$ proving that $2^{2/3}$ is rational and that $2^{1/3}$ is rational only if $2^{2/3}$ is rational.

A1 (4)

Assume that $2^{1/3}$ is rational.

Define the set T to be the set of positive integers with the following property: n is in T if and only if $n2^{1/3}$ and $n2^{2/3}$ are integers. **B1**

The set T contains at least one positive integer as if $2^{1/3} = {}^{p}/_{q}$, where $p, q \in \mathbb{Z}, q \neq 0$, then $q^2 2^{1/3} = q^2 \times {}^{p}/_{q} = pq \in \mathbb{Z}$ and $q^2 2^{2/3} = q^2 \times {}^{p^2}/_{q^2} = p^2 \in \mathbb{Z}$, so $q^2 \in T$. **M1A1**

Define t to be the smallest positive integer in T. Then $t2^{1/3}$ and $t2^{2/3}$ are integers. **B1**

Consider $t\left(2^{2/3} - 1\right)$. $t\left(2^{2/3} - 1\right) = t2^{2/3} - t$ which is the difference of two integers and so is itself an integer. $t\left(2^{2/3} - 1\right) \times 2^{1/3} = 2t - t2^{1/3}$ which is an integer,

and $t\left(2^{2/3} - 1\right) \times 2^{2/3} = 2^{4/3}t - t2^{2/3} = 2 \times 2^{1/3}t - t2^{2/3}$ which is an integer.

Thus $t\left(2^{2/3} - 1\right)$ is in T. **M1A1**

$1 < 2^{2/3} < 2$ and so $0 < 2^{2/3} - 1 < 1$, and thus $0 < t\left(2^{2/3} - 1\right) < t$, and thus this contradicts that t is the smallest positive integer in T as $t\left(2^{2/3} - 1\right)$ has been shown to be a smaller positive integer and is in T . **M1A1 (8)**