STEP III, 2015 , Q4 MS

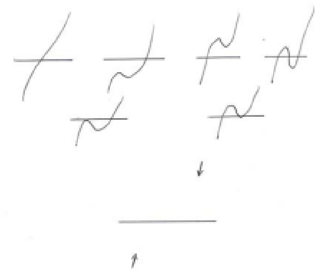
4. Part (i) is imply shown by considering the image of the function $f(z) = z^3 + az^2 + bz + c$ as $z \to \pm\infty$ and then observing that the function is continuous and exhibits a sign-change. Part (ii) can be approached by writing $z^3 + az^2 + bz + c = (z - z_1)(z - z_2)(z - z_3)$ giving $a = -S_1$, $= \frac{S_1{}^2 - S_2}{2}$, which can be obtained by considering $(z_1 + z_2 + z_3)^2$ and the required result for $6c$ which can be neatly obtained by considering $f(z_1) + f(z_2) + f(z_3) = 0$.

Writing $z_k = r_k(\cos\theta_k + i\sin\theta_k)$ for $k = 1, 2, 3$, employing de Moivre's theorem, the three sums imply the reality of S_1 , S_2 , and S_3 , and hence a , b , and c which by virtue of the result of part (i) yields the reality of z_1 , z_2 , or z_3 and hence the required result. The final result can be considered as two cases, the trivial one of all three roots being real, and the one where the other two are complex. The latter can be shown to give the required result by considering the real and imaginary parts of the roots of a real quadratic.

4. (i) $y = z^3 + az^2 + bz + c$ is continuous.

For $z \to -\infty$, $y \to -\infty$ and for $z \to \infty$, $y \to \infty$. **B1**

So the sketch of this graph must be one of the following:-



B1

Hence, it must intersect the z axis at least once, and so there is at least one real root of

$z^3 + az^2 + bz + c = 0$ **B1 (3)**

(ii) $z^3 + az^2 + bz + c = (z - z_1)(z - z_2)(z - z_3)$ **M1**

Thus $a = (-z_1 - z_2 - z_3) = -S_1$ **A1**

$b = (z_2 z_3 + z_3 z_1 + z_1 z_2) = \frac{(z_1+z_2+z_3)^2 - (z_1{}^2+z_2{}^2+z_3{}^2)}{2} = \frac{S_1{}^2 - S_2}{2}$ **A1**

and, as $z_1{}^3 + az_1{}^2 + bz_1 + c = 0$, $z_2{}^3 + az_2{}^2 + bz_2 + c = 0$, $z_3{}^3 + az_3{}^2 + bz_3 + c = 0$

adding these three equations we have,

$(z_1{}^3 + z_2{}^3 + z_3{}^3) + a(z_1{}^2 + z_2{}^2 + z_3{}^2) + b(z_1 + z_2 + z_3) + 3c = 0$ **M1**

(Alternatively,

$(z_1 + z_2 + z_3)^3 =$

$$(z_1{}^3 + z_2{}^3 + z_3{}^3) + 3(z_1{}^2 z_2 + z_2{}^2 z_3 + z_3{}^2 z_1 + z_1{}^2 z_3 + z_2{}^2 z_1 + z_3{}^2 z_2) + 6z_1 z_2 z_3$$

$(z_1{}^2 + z_2{}^2 + z_3{}^2)(z_1 + z_2 + z_3) = (z_1{}^3 + z_2{}^3 + z_3{}^3) + (z_1{}^2 z_2 + z_2{}^2 z_3 + z_3{}^2 z_1 + z_1{}^2 z_3 + z_2{}^2 z_1 + z_3{}^2 z_2)$)

So $S_3 - S_1 S_2 + \frac{S_1{}^2 - S_2}{2} S_1 + 3c = 0$ **M1**

Thus $6c = \left(3S_1 S_2 - S_1{}^3 - 2S_3\right)$ **A1* (6)**

(iii) Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$ for $k = 1, 2, 3$ **M1**

Then $z_k{}^2 = r_k{}^2(\cos 2\theta_k + i\sin 2\theta_k)$ and $z_k{}^3 = r_k{}^3(\cos 3\theta_k + i\sin 3\theta_k)$ by de Moivre **M1**

As

$$\sum_{k=1}^{3} r_k \sin\theta_k = 0$$

$$\sum_{k=1}^{3} r_k{}^2 \sin 2\theta_k = 0$$

$$\sum_{k=1}^{3} r_k{}^3 \sin 3\theta_k = 0$$

$$Im\left(\sum_{k=1}^{3} z_k\right) = 0$$

$$Im\left(\sum_{k=1}^{3} z_k{}^2\right) = 0$$

$$Im\left(\sum_{k=1}^{3} z_k{}^3\right) = 0$$

and so S_1, S_2, and S_3 are real, **M1**

and therefore so are a, b, and c **A1**

Hence, as z_1, z_2, and z_3 are the roots of $z^3 + az^2 + bz + c = 0$ with a, b, and c real, by part (i), at least one of z_1, z_2, and z_3 is real. **M1**

So for at least one value of k, $r_k(\cos\theta_k + i\sin\theta_k)$ is real and thus, $\sin\theta_k = 0$,

and as $-\pi < \theta_k < \pi$, $\theta_k = 0$ as required. **A1 (6)**

If $\theta_1 = 0$ then z_1 is real. z_2 and z_3 are the roots of $(z - z_2)(z - z_3) = 0$

which is $z^2 + (-z_2 - z_3)z + z_2 z_3 = 0$ (say $z^2 + pz + q = 0$)

$p = -z_2 - z_3 = a + z_1$ and $q = z_2 z_3 = -\frac{c}{z_1}$ and so the quadratic of which z_2 and z_3 are the roots has real coefficients. Thus $z_2, z_3 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$. ($z_1 \neq 0$ because $r_k > 0$) **B1**

If $p^2 - 4q < 0$, **M1**

Thus $\cos\theta_2 = \cos\theta_3$, and so $\theta_2 = \pm\theta_3$, as $-\pi < \theta_k < \pi$.

But $\sin\theta_2 = -\sin\theta_3$ and so $\theta_2 = -\theta_3$. **M1 A1**

If $p^2 - 4q \geq 0$, then z_2 and z_3 are real roots, so $\sin\theta_2 = \sin\theta_3 = 0$, and thus $\theta_2 = \theta_3 = 0$, so $\theta_2 = -\theta_3$. **B1 (5)**