STEP III, 2015 , Q4

4 **(i)** If a, b and c are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

has at least one real root.

(ii) Let

$$S_1 = z_1 + z_2 + z_3, \quad S_2 = z_1^2 + z_2^2 + z_3^2, \quad S_3 = z_1^3 + z_2^3 + z_3^3\,,$$

where z_1, z_2 and z_3 are the roots of the equation $(*)$. Express a and b in terms of S_1 and S_2, and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3\,.$$

(iii) The six real numbers r_k and θ_k $(k = 1,\ 2,\ 3)$, where $r_k > 0$ and $-\pi < \theta_k < \pi$, satisfy

$$\sum_{k=1}^{3} r_k \sin(\theta_k) = 0\,, \qquad \sum_{k=1}^{3} r_k^2 \sin(2\theta_k) = 0\,, \qquad \sum_{k=1}^{3} r_k^3 \sin(3\theta_k) = 0\,.$$

Show that $\theta_k = 0$ for at least one value of k.

Show further that if $\theta_1 = 0$ then $\theta_2 = -\theta_3$.