STEP III, 2015 , Q2 MS

2. Part (ii) is the only false statement, and a simple counter-example is $s_n = 1$ and $t_n = 2$ for n odd, and $s_n = 2$ and $t_n = 1$ for n even. Part (i) $m = 1000$ is a suitable value, then $1000 \leq n$ and as n is positive, the inequality can be multiplied by it giving the required result. Part (iii) requires the use of the definition twice with values m_1 and m_2 say, and then using $m = \max(m_1, m_2)$. For part (iv), we can choose $m = 4$, and an inductive argument such as

$$(k+1)^2 = \left(1+\frac{1}{k}\right)^2 k^2 \leq \left(1+\frac{1}{4}\right)^2 k^2 < 2k^2 \leq 2 \times 2^k = 2^{k+1}$$ works.

2. (i) True. **B1**

$m = 1000$ **B1**

If $n \geq 1000$, then $1000 \leq n$, so $1000n \leq n^2$, i.e. $(1000n) \leq (n^2)$ **M1A1 (4)**

(ii) False. **B1**

E. G. Let $s_n = 1$ and $t_n = 2$ for n odd, and $s_n = 2$ and $t_n = 1$ for n even. **B1**

Then $\nexists m$ for which for $n \geq m$, $s_n \leq t_n$, nor $t_n \leq s_n$ **M1**

So it is not the case that $(s_n) \leq (t_n)$, but nor is it the case that $(t_n) \leq (s_n)$ **A1 (4)**

(iii) True. **B1**

$(s_n) \leq (t_n)$ means that there exists a positive integer, say m_1, for which for $n \geq m_1$, $s_n \leq t_n$.
E1

$(t_n) \leq (u_n)$ means that there exists a positive integer, say m_2, for which for $n \geq m_2$, $t_n \leq u_n$.
E1

Then if $m = \max(m_1, m_2)$, **B1**

for $n \geq m$, $s_n \leq t_n \leq u_n$, and so $(s_n) \leq (u_n)$ **A1 (5)**

(iv) True. **B1**

$m = 4$ **B1**

Assume $k^2 \leq 2^k$ for some value $k \geq 4$. **B1**

Then $(k+1)^2 = \left(\frac{k+1}{k}\right)^2 k^2 = \left(1+\frac{1}{k}\right)^2 k^2 \leq \left(1+\frac{1}{4}\right)^2 k^2 = \frac{25}{16}k^2 < 2k^2 \leq 2 \times 2^k = 2^{k+1}$
M1A1

$4^2 = 2^4$ **B1**

so by the principle of mathematical induction, $n^2 \leq 2^n$ for $n \geq 4$, and thus $(n^2) \leq (2^n)$ **A1 (7)**