STEP III, 2015 , Q1 MS

1. The first result can be obtained by simplifying the LHS and then writing it as $\int_0^\infty u \frac{u}{(1+u^2)^{n+1}} du$ and integrating this by parts. To obtain the evaluation of I_{n+1}, the first result can be re-arranged to make I_{n+1} the subject, and then iterating the result to express it in terms of I_1 which is a standard integral. The expression can be tidied by multiplying numerator and denominator by $(2n)(2n-2) \ldots (2)$. The first result for (ii) is obtained by means of the substitution $u = x^{-1}$, the second by adding the two versions of J, and the third by the substitution $u = x - x^{-1}$, being careful with limits of integration and employing symmetry. Part (iii) is solved by expressing the integrand as $\frac{x^{-2}}{((x-x^{-1})^2+1)^n}$ and then employing first part (ii) then part (i) to obtain I_n, which is $\frac{(2n-2)!\pi}{2^{2n-1}((n-1)!)^2}$.

1. (i)

$$I_n - I_{n+1} = \int_0^\infty \frac{1}{(1+u^2)^n} du - \int_0^\infty \frac{1}{(1+u^2)^{n+1}} du = \int_0^\infty \frac{1+u^2-1}{(1+u^2)^{n+1}} du$$

$$= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}} du$$

B1

$$= \int_0^\infty u \frac{u}{(1+u^2)^{n+1}} du = \left[u \frac{-1}{2n(1+u^2)^n} \right]_0^\infty - \int_0^\infty \frac{-1}{2n(1+u^2)^n} du$$

integrating by parts M1 A1

$$= 0 + \frac{1}{2n} \int_0^\infty \frac{1}{(1+u^2)^n} du = \frac{1}{2n} I_n$$

A1* (4)

$I_{n+1} = I_n - \frac{1}{2n} I_n = \frac{2n-1}{2n} I_n$ M1

$= \frac{(2n-1)(2n-3)\ldots(1)}{(2n)(2n-2)\ldots(2)} I_1$ M1

$I_1 = \int_0^\infty \frac{1}{(1+u^2)} du = [\tan^{-1} u]_0^\infty = \frac{\pi}{2}$ B1

$\frac{(2n-1)(2n-3)\ldots(1)}{(2n)(2n-2)\ldots(2)} = \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots(2)(1)}{[(2n)(2n-2)\ldots(2)]^2} = \frac{(2n)!}{[2^n n!]^2} = \frac{(2n)!}{2^{2n}(n!)^2}$ M1

Thus $I_{n+1} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\pi}{2} = \frac{(2n)!\pi}{2^{2n+1}(n!)^2}$ A1* (5)

(ii)

$$J=\int_0^\infty f((x-x^{-1})^2)dx=\int_\infty^0 f((u^{-1}-u)^2).-u^{-2}du=\int_0^\infty x^{-2}f((x-x^{-1})^2)dx$$

using the substitution $u=x^{-1}$, $\frac{du}{dx}=-x^{-2}$ and then the substitution $u=x$, $\frac{du}{dx}=1$ **M1A1***

$$2J=\int_0^\infty f((x-x^{-1})^2)dx+\int_0^\infty x^{-2}f((x-x^{-1})^2)dx=\int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx$$

So $J=\frac{1}{2}\int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx$ **M1A1***

Using the substitution $u=x-x^{-1}$, $\frac{du}{dx}=1+x^{-2}$,

$J=\frac{1}{2}\int_0^\infty f((x-x^{-1})^2)(1+x^{-2})dx=\frac{1}{2}\int_{-\infty}^\infty f(u^2)\,du=\int_0^\infty f(u^2)du$ M1A1* (6)

(iii)

$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}dx=\int_0^\infty \frac{x^{-2}}{(x^2-1+x^{-2})^n}dx=\int_0^\infty \frac{x^{-2}}{((x-x^{-1})^2+1)^n}dx$ **M1A1**

$=\int_0^\infty \frac{1}{(u^2+1)^n}du$ **M1**

So

$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}dx=\int_0^\infty \frac{1}{(u^2+1)^n}du=I_n$ **M1**

$=\frac{(2(n-1))!\pi}{2^{2(n-1)+1}((n-1)!)^2}=\frac{(2n-2)!\pi}{2^{2n-1}((n-1)!)^2}$ **A1 (5)**