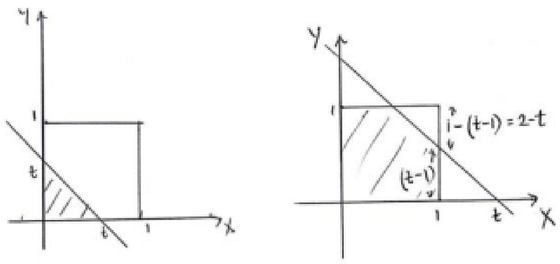
STEP III, 2015 , Q13 MS

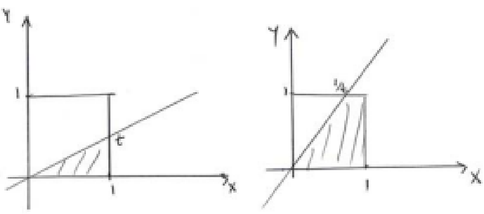
13. (i)



lead to $P(X+Y<t) = \frac{1}{2}t^2$ if $0 \le t \le 1$, $P(X+Y<t) = 1 - \frac{1}{2}(2-t)^2$ if $1 < t \le 2$, $P(X+Y<t) = 0$ if $t<0$ and $P(X+Y<t) = 1$ if $t>2$.

From this, the cumulative distribution function of $(X+Y)^{-1}$ by means of the logic $P((X+Y)^{-1} < t) = P\left(X+Y > \frac{1}{t}\right) = 1 - P\left(X+Y < \frac{1}{t}\right)$ and the required probability density function can be found by differentiation. From that, by integration, $E\left(\frac{1}{X+Y}\right) = 2\ln 2$

(ii)



lead to

$$P\left(\frac{Y}{X} < t\right) = \begin{cases} \frac{1}{2}t \ \ for\ 0 \le t \le 1 \\ 1 - \frac{1}{2}t^{-1} \ \ for\ t > 1 \end{cases}$$

That $P\left(\frac{X}{X+Y} < t\right) = P\left(\frac{X+Y}{X} > \frac{1}{t}\right) = P\left(\frac{Y}{X} > \frac{1}{t} - 1\right) = 1 - P\left(\frac{Y}{X} < \frac{1}{t} - 1\right)$ and differentiation leads to

the probability density function $f(t) = \begin{cases} \frac{1}{2}t^{-2} \ \ for\ \frac{1}{2} \le t \le 1 \\ \frac{1}{2}(1-t)^{-2} \ \ for\ 0 \le t < \frac{1}{2} \end{cases}$.

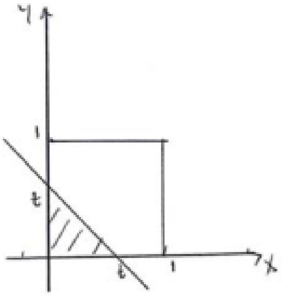
$E\left(\frac{X}{X+Y}\right) = \frac{1}{2}$ can be written down because by symmetry, $E\left(\frac{X}{X+Y}\right) = E\left(\frac{Y}{X+Y}\right)$

and $E\left(\frac{X}{X+Y}\right) + E\left(\frac{Y}{X+Y}\right) = E\left(\frac{X+Y}{X+Y}\right) = E(1) = 1$. This is simply verified by integration using the probability density function found.

[An earlier potential version of the question had X , Y , and Z independently uniformly distributed on $[0,1]$, considered the distribution of $\ln X$,

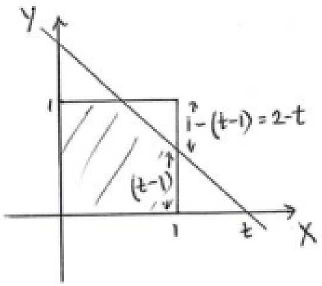
went on to find the pdf of U , where $U = -\ln(XY)$ and finished by showing that $(XY)^Z$ is also uniformly distributed on $[0,1]$.]

13. (i)



G1

$P(X + Y < t) = \frac{1}{2}t^2$ if $0 \le t \le 1$ **B1**



G1

and $P(X + Y < t) = 1 - \frac{1}{2}(2 - t)^2$ if $1 < t \le 2$ **B1**

$P(X + Y < t) = 0$ if $t < 0$ and $P(X + Y < t) = 1$ if $t > 2$

So $F(t) = \begin{cases} 0 \quad for\, t < 0 \\ \frac{1}{2}t^2 \quad for\, 0 \le t \le 1 \\ 1 - \frac{1}{2}(2 - t)^2 \quad for\, 1 < t \le 2 \\ 1\, for\, t > 2 \end{cases}$

B1 (5)

Thus $P((X+Y)^{-1} < t) = P\left(X+Y > \frac{1}{t}\right) = 1 - P\left(X+Y < \frac{1}{t}\right)$

$$= \begin{cases} 1 - \frac{1}{2t^2} \ for\ 1 \le t \\ \frac{1}{2}\left(2 - \frac{1}{t}\right)^2 \ for\ \frac{1}{2} \le t < 1 \\ 0 \ for\ t < \frac{1}{2} \end{cases}$$

M1 A1

So as $f(t) = \frac{dF(t)}{dt}$,

$$f(t) = \begin{cases} 0 \ for\ t < \frac{1}{2} \\ \frac{1}{t^2}\left(2 - \frac{1}{t}\right) \ for\ \frac{1}{2} \le t < 1 \\ \frac{1}{t^3} \ for\ 1 \le t \end{cases}$$

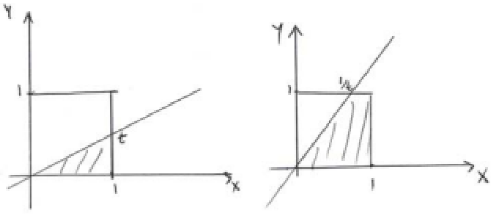
as required. **M1A1* (4)**

$$E\left(\frac{1}{X+Y}\right) = \int_{\frac{1}{2}}^{1} t(2t^{-2} - t^{-3})dt + \int_{1}^{\infty} t.t^{-3}dt = [2\ln t + t^{-1}]_{\frac{1}{2}}^{1} + [-t^{-1}]_{1}^{\infty}$$

$$= 1 - 2\ln\frac{1}{2} - 2 + 1 = 2\ln 2$$

M1 A1 (2)

(ii)



G1

$$P\left(\frac{Y}{X} < t\right) = \begin{cases} \frac{1}{2}t \ \ for\ 0 \le t \le 1 \\ 1 - \frac{1}{2}t^{-1} \ \ for\ t > 1 \end{cases}$$

B1 (2)

Thus

$$P\left(\frac{X}{X+Y} < t\right) = P\left(\frac{X+Y}{X} > \frac{1}{t}\right) = P\left(1 + \frac{Y}{X} > \frac{1}{t}\right) = P\left(\frac{Y}{X} > \frac{1}{t} - 1\right) = 1 - P\left(\frac{Y}{X} < \frac{1}{t} - 1\right)$$

So

$$F(t) = \begin{cases} 1 - \frac{1}{2}\left(\frac{1}{t} - 1\right) \ \ for\ \frac{1}{2} \le t \le 1 \\ \frac{1}{2}\left(\frac{1}{t} - 1\right)^{-1} \ \ for\ 0 \le t < \frac{1}{2} \end{cases}$$

i.e.

$$F(t) = \begin{cases} \frac{1}{2}(3 - t^{-1}) \ \ for\ \frac{1}{2} \le t \le 1 \\ \frac{1}{2}\left(\frac{t}{1-t}\right) \ \ for\ 0 \le t < \frac{1}{2} \end{cases}$$

M1A1

So as $f(t) = \frac{dF(t)}{dt}$,

$$f(t) = \begin{cases} \frac{1}{2}t^{-2} & for\ \frac{1}{2} \le t \le 1 \\ \frac{1}{2}(1-t)^{-2} & for\ 0 \le t < \frac{1}{2} \end{cases}$$

M1A1 (4)

$E\left(\frac{X}{X+Y}\right) = \frac{1}{2}$ because, by symmetry, $E\left(\frac{X}{X+Y}\right) = E\left(\frac{Y}{X+Y}\right)$

and $E\left(\frac{X}{X+Y}\right) + E\left(\frac{Y}{X+Y}\right) = E\left(\frac{X+Y}{X+Y}\right) = E(1) = 1$ B1

$$E\left(\frac{X}{X+Y}\right) = \int_0^{\frac{1}{2}} \frac{1}{2}t(1-t)^{-2}\,dt + \int_{\frac{1}{2}}^{1} t \times \frac{1}{2}t^{-2}\,dt$$

$$= \left[\frac{1}{2}t(1-t)^{-1}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{1}{2}(1-t)^{-1}dt + \left[\frac{1}{2}\ln t\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{2} - \left[-\frac{1}{2}\ln(1-t)\right]_0^{\frac{1}{2}} - \frac{1}{2}\ln\frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}$$

as required. M1A1 (3)