STEP III, 2015 , Q12 MS

12. (i) The required probability generating function is $G(x) = \frac{1}{6}(1 + x + x^2 + x^3 + x^4 + x^5)$ and it is simple to write down the probability distribution function of S_2 and hence of R_2 and arrive at the same pgf. As a consequence, it can be argued that the pgf for R_n is also $G(x)$ and so the required probability is $1/6$.

(ii) $G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4) = \frac{1}{6}(x + y)$. $G_2(x)$ would be $(G_1(x))^2$ except that the powers must be multiplied congruent to modulus 5 , and it can be shown that $xy = y$ and $y^2 = 5y$ so obtaining the required result for $G_2(x)$. Obtaining $G_n(x) = \frac{1}{6^n}\left(x^{n-5p} + \frac{6^n - 1}{5}y\right)$

where p is an integer such that $0 \le n - 5p \le 4$, and the probability that S_n is divisible by 5 will be the coefficient of x^0 which in turn is the coefficient of y as required. If n is divisible by 5, the probability that S_n is divisible by 5 will be $\frac{1}{5}\left(1 + \frac{4}{6^n}\right)$ as $x^{n-5p} = x^0$.

12. (i) The probability distribution function of S_1 is

S_1	1	2	3	4	5	6
p	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

so the probability distribution function of R_1 is

R_1	0	1	2	3	4	5
p	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

and thus $G(x) = \frac{1}{6}(1 + t + t^2 + t^3 + t^4 + t^5)$. **B1**

The probability distribution function of S_2 is

S_2	2	3	4	5	6	7	8	9	10	11	12
p	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

M1

so the probability distribution function of R_2 is

R_2	0	1	2	3	4	5
p	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$	$6/36$

A1

which is the same as for R_1 and hence its probability generating function is also $G(x)$. **A1***

Therefore, the probability generating function of R_n is also $G(x)$ **B1**

and thus the probability that S_n is divisible by 6 is ${}^1/_6$. **B1 (6)**

(ii) The probability distribution function of T_1 is

T_1	0	1	2	3	4
p	${}^1/_6$	${}^2/_6$	${}^1/_6$	${}^1/_6$	${}^1/_6$

and thus $G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4)$ **M1 A1**

$G_2(x)$ would be $\left(G_1(x)\right)^2$ except that the powers must be multiplied congruent to modulus 5.

$G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4) = \frac{1}{6}(x + 1 + x + x^2 + x^3 + x^4) = \frac{1}{6}(x + y)$ **B1**

Thus $G_2(x)$ would be $\frac{1}{36}(x + y)^2$

except $xy = x(1 + x + x^2 + x^3 + x^4) = x + x^2 + x^3 + x^4 + 1 = y$ **M1A1**

and $y^2 = (1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4) = (1 + x + x^2 + x^3 + x^4) + (x + x^2 + x^3 + x^4 + 1) + (x^2 + x^3 + x^4 + 1 + x) + (x^3 + x^4 + 1 + x + x^2) + (x^4 + 1 + x + x^2 + x^3) = 5y$ **A1**

So $G_2(x) = \frac{1}{36}(x + y)^2 = \frac{1}{36}(x^2 + 2xy + y^2) = \frac{1}{36}(x^2 + 2y + 5y) = \frac{1}{36}(x^2 + 7y)$ **M1A1* (8)**

$$G_3(x) = \frac{1}{6^3}(x + y)^3 = \frac{1}{6^3}(x + y)(x^2 + 7y) = \frac{1}{6^3}(x^3 + yx^2 + 7xy + 7y^2)$$

That is

$$G_3(x) = \frac{1}{6^3}(x^3 + yx^2 + 7xy + 7y^2) = \frac{1}{6^3}(x^3 + y + 7y + 35y) = \frac{1}{6^3}(x^3 + 43y)$$

We notice that the coefficient of y inside the bracket in $G_n(x)$ is $(1 + 6 + 6^2 + \cdots 6^{n-1})$

This can be shown simply by induction. It is true for $n = 1$ trivially.

Consider $(x+y)(x^r+(1+6+6^2+\cdots+6^{k-1})y) = x^{r+1}+yx^r+(1+6+6^2+\cdots+6^{k-1})xy+(1+6+6^2+\cdots+6^{k-1})y^2$

$$yx^r+(1+6+6^2+\cdots 6^{k-1})xy+(1+6+6^2+\cdots+6^{k-1})y^2$$
$$= y+(1+6+6^2+\cdots+6^{k-1})y+5(1+6+6^2+\cdots+6^{k-1})y$$

$$5(1+6+6^2+\cdots 6^{k-1}) = (6-1)(1+6+6^2+\cdots 6^{k-1}) = 6^k-1$$

So $y+(1+6+6^2+\cdots+6^{k-1})y+5(1+6+6^2+\cdots+6^{k-1})y = (1+6+6^2+\cdots 6^k)y$

as required. M1

However, this coefficient is the sum of a GP and so $G_n(x) = \frac{1}{6^n}\left(x^{n-5p}+\frac{6^n-1}{5}y\right)$ where p is an integer such that $0 \le n-5p \le 4$. M1 A1

So if n is not divisible by 5, the probability that S_n is divisible by 5 will be the coefficient of x^0 which in turn is the coefficient of y, namely $\frac{1}{6^n}\left(\frac{6^n-1}{5}\right) = \frac{1}{5}\left(1-\frac{1}{6^n}\right)$ as required. B1*

If n is divisible by 5, the probability that S_n is divisible by 5 will be $\frac{1}{6^n}\left(1+\frac{6^n-1}{5}\right)$ as $x^{n-5p} = x^0$

That is $\frac{1}{5}\left(1+\frac{4}{6^n}\right)$ M1A1 (6)