STEP III, 2014 , Q9

9 A particle of mass m is projected with velocity $\mathbf{u}$. It is acted upon by the force $m\mathbf{g}$ due to gravity and by a resistive force $-mk\mathbf{v}$, where $\mathbf{v}$ is its velocity and k is a positive constant.

Given that, at time t after projection, its position $\mathbf{r}$ relative to the point of projection is given by

$$\mathbf{r} = \frac{kt - 1 + \mathrm{e}^{-kt}}{k^2}\,\mathbf{g} + \frac{1 - \mathrm{e}^{-kt}}{k}\,\mathbf{u}\,,$$

find an expression for $\mathbf{v}$ in terms of k, t, $\mathbf{g}$ and $\mathbf{u}$. Verify that the equation of motion and the initial conditions are satisfied.

Let $\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$ and $\mathbf{g} = -g\,\mathbf{j}$, where $0 < \alpha < 90^\circ$, and let T be the time after projection at which $\mathbf{r}\,.\,\mathbf{j} = 0$. Show that

$$uk\sin\alpha = \left(\frac{kT}{1 - \mathrm{e}^{-kT}} - 1\right) g\,.$$

Let β be the acute angle between $\mathbf{v}$ and $\mathbf{i}$ at time T. Show that

$$\tan\beta = \frac{(\mathrm{e}^{kT} - 1)g}{uk\cos\alpha} - \tan\alpha\,.$$

Show further that $\tan\beta > \tan\alpha$ (you may assume that $\sinh kT > kT$) and deduce that $\beta > \alpha$.