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7. Part (i), the intersecting chords theorem, is basic bookwork relying on angle properties in circles to establish similar triangles and hence the result. Part (ii) can be obtained by considering that Q lies on P_1P_3 and so $\boldsymbol{q} = \boldsymbol{p_1} + \lambda(\boldsymbol{p_3} - \boldsymbol{p_1})$, that Q also lies on P_2P_4 producing a similar result and then equating these two expressions, finally rearranging to give (*). Assuming that $a_1 + a_3 = 0$ and using (*) leads to $a_1(\boldsymbol{p_1} - \boldsymbol{p_3}) = a_2(\boldsymbol{p_4} - \boldsymbol{p_2})$ which, in view of the distinctness of the four points P and the intersection of P_1P_3 and P_2P_4 at Q , leads to the contradiction $a_1 = a_2 = a_3 = a_4 = 0$. Re-writing $\frac{a_1\boldsymbol{p_1}+a_3\boldsymbol{p_3}}{a_1+a_3}$ as $\boldsymbol{p_1} + \frac{a_3(\boldsymbol{p_3}-\boldsymbol{p_1})}{a_1+a_3}$ and similarly, using (*), as $\frac{a_2\boldsymbol{p_2}+a_4\boldsymbol{p_4}}{a_2+a_4}$ and re-writing, the expression can be shown to be the position vector of Q .The final result comes from applying (i) using the information just gained and calculating both expressions by taking scalar products of the vectors whose magnitudes are quoted in (i).