STEP III, 2014 , Q7

7 The four distinct points P_i $(i = 1, 2, 3, 4)$ are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines P_1P_3 and P_2P_4 intersect at Q.

(i) By considering the triangles P_1QP_4 and P_2QP_3 show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)$.

(ii) Let $\mathbf{p}_i$ be the position vector of the point P_i $(i = 1, 2, 3, 4)$. Show that there exist numbers a_i, not all zero, such that

$$\sum_{i=1}^{4} a_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} a_i \mathbf{p}_i = \mathbf{0}\,. \qquad (*)$$

(iii) Let a_i $(i = 1, 2, 3, 4)$ be any numbers, not all zero, that satisfy $(*)$. Show that $a_1 + a_3 \neq 0$ and that the lines P_1P_3 and P_2P_4 intersect at the point with position vector

$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}\,.$$

Deduce that $a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$.