STEP III, 2014 , Q6 MS

6. Starting from $f''(t) > 0$ for $0 < t < x_0 \implies \int_0^{t_0} f''(t)\,dt > 0$ where $0 < t_0 < x_0$, with the given conditions yields $f'(t_0) > 0$, and then repeating the argument with $f'(t)$ instead gives $f(t) > 0$. Choosing $f(x) = 1 - \cos x \cosh x$ and applying the applying the stem of the question for $0 < x < \pi$, gives the required inequality for $0 < x < \pi/2$ in particular. For part (ii), choosing $g(x) = x^2 - \sin x \sinh x$ (in which case $g''(x) = 2f(x)$, where $f(x)$ was the suggested choice for part (i)) and $h(x) = \sin x \cosh x - x$ provide the desired results once care is taken with positivity of functions over the required interval when dividing inequalities.