STEP III 2014 Question 5 — Mark Scheme

ABCD is a parallelogram if and only if $\overrightarrow{AB} = \overrightarrow{DC}$, **B1**
i.e. if and only if $b - a = c - d$ **M1**
which rearranged gives $a + c = b + d$ as required. **(*) A1** **(3)**
Further, to be a square as well, angle $ABC = 90°$, and $|AB| = |BC|$. **M1**
Thus $|c - b| = |b - a|$ and $(c - b) = arg(b - a) + 90°$, so $c - b = i(b - a)$. **B1**
So $a + c = b + d$ and $ia + c = (i + 1)b$, and thus $a(1 - i) = d - ib$ yielding
$a(1 - i)(1 + i) = (d - ib)(1 + i)$, hence $a = \frac{1}{2}\big((1 - i)b + (1 + i)d\big)$ and so
$c = b + d - \frac{1}{2}\big((1 - i)b + (1 + i)d\big) = \frac{1}{2}\big((1 + i)b + (1 - i)d\big)$
So $i(a - c) = i\left[\frac{1}{2}\big((1 - i)b + (1 + i)d\big) - \frac{1}{2}\big((1 + i)b + (1 - i)d\big)\right] = i[-ib + id] = b - d$ **B1 (3)**
(alternatively, ABCD square $\Longleftrightarrow$ ABCD parallelogram **B1** & diagonals equal length and perpendicular **B1** $\Longleftrightarrow i(a - c) = b - d$ **B1 (3)**)
(i) angle $PXQ = 90°$, and $|PX| = |XQ|$ so, replicating result with ABC in stem, **M1**
$q - x = i(x - p)$ **A1** so $(1 + i)x = ip + q$, $(1 - i)(1 + i)x = (1 - i)(ip + q)$, **M1** and hence
$2x = (1 + i)p + (1 - i)q$ and $x = \frac{1}{2}\big(p(1 + i) + q(1 - i)\big)$ **A1 (4)**
(ii) From the stem, XYZT is a square if and only if $i(x - z) = y - t$, and $x + z = y + t$ **B1 B1**
$\Longleftrightarrow i\left(\frac{1}{2}\big(p(1 + i) + q(1 - i)\big) - \frac{1}{2}\big(r(1 + i) + s(1 - i)\big)\right) = \frac{1}{2}\big(q(1 + i) + r(1 - i)\big) - \frac{1}{2}\big(s(1 + i) + p(1 - i)\big)$ **M1 A1 A1**
$\Longleftrightarrow p(i - 1) + q(1 + i) + r(1 - i) - s(1 + i) = p(i - 1)i + q(1 + i) + r(1 - i) - s(1 + i)$ which is trivially true **M1A1** **(7)**
and
$\frac{1}{2}\big(p(1 + i) + q(1 - i)\big) + \frac{1}{2}\big(r(1 + i) + s(1 - i)\big) = \frac{1}{2}\big(q(1 + i) + r(1 - i)\big) + \frac{1}{2}\big(s(1 + i) + p(1 - i)\big)$
M1 A1
$\Longleftrightarrow ip - iq + ir - is = 0 \Longleftrightarrow p + r = q + s$ so PQRS is a parallelogram. **B1 (3)**

SCAN OR CLICK QUESTION
SCAN OR CLICK EXAMINER'S COMMENTS