STEP III, 2014 , Q5

5 A quadrilateral drawn in the complex plane has vertices A, B, C and D, labelled anticlockwise. These vertices are represented, respectively, by the complex numbers a, b, c and d. Show that $ABCD$ is a parallelogram (defined as a quadrilateral in which opposite sides are parallel and equal in length) if and only if $a + c = b + d$. Show further that, in this case, $ABCD$ is a square if and only if $\mathrm{i}(a - c) = b - d$.

Let $PQRS$ be a quadrilateral in the complex plane, with vertices labelled anticlockwise, the internal angles of which are all less than $180°$. Squares with centres X, Y, Z and T are constructed externally to the quadrilateral on the sides PQ, QR, RS and SP, respectively.

(i) If P and Q are represented by the complex numbers p and q, respectively, show that X can be represented by

$$\tfrac{1}{2}\big(p(1+\mathrm{i}) + q(1-\mathrm{i})\big)\,.$$

(ii) Show that $XYZT$ is a square if and only if $PQRS$ is a parallelogram.