STEP III, 2014 , Q4

4 (i) Let

$$I = \int_0^1 \left((y')^2 - y^2\right) \mathrm{d}x \qquad \text{and} \qquad I_1 = \int_0^1 (y' + y\tan x)^2 \, \mathrm{d}x \,,$$

where y is a given function of x satisfying $y = 0$ at $x = 1$. Show that $I - I_1 = 0$ and deduce that $I \geqslant 0$. Show further that $I = 0$ only if $y = 0$ for all x $(0 \leqslant x \leqslant 1)$.

(ii) Let

$$J = \int_0^1 \left((y')^2 \quad a^2y^2\right) \mathrm{d}x \,,$$

where a is a given positive constant and y is a given function of x, not identically zero, satisfying $y = 0$ at $x = 1$. By considering an integral of the form

$$\int_0^1 (y' + ay\tan bx)^2 \, \mathrm{d}x \,,$$

where b is suitably chosen, show that $J \geqslant 0$. You should state the range of values of a, in the form $a < k$, for which your proof is valid.

In the case $a = k$, find a function y (not everywhere zero) such that $J = 0$.