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12. The first result, $y_m = e^{x_m}$, is obtained merely by considering probabilities, and the given pdf of Y can be obtained by standard techniques or by consideration of changing the variable in the integral of the pdf of X. The mode result relies on differentiation of the pdf of Y equated to zero to give a stationary value. The explanation in part (iii) is simply that the required integral is merely that of the pdf of a Normal variable with mean $\mu + \sigma^2$. The expectation of Y is obtained in the standard manner, using an integral and the pdf of Y, and then a change of variable, in which exponential terms can be combined so as to use the explained result having completed the square in the exponent. Using the three previous parts gives $\lambda = e^{\mu-\sigma^2}$, $y_m = e^{x_m} = e^{\mu}$ because X is symmetric, and, as stated,

$E(Y) = e^{\mu+\frac{1}{2}\sigma^2}$, hence satisfying part (iv).