STEP III, 2014 , Q12

12 The random variable X has probability density function $\mathrm{f}(x)$ (which you may assume is differentiable) and cumulative distribution function $\mathrm{F}(x)$ where $-\infty < x < \infty$. The random variable Y is defined by $Y = \mathrm{e}^X$. You may assume throughout this question that X and Y have unique modes.

(i) Find the median value y_m of Y in terms of the median value x_m of X.

(ii) Show that the probability density function of Y is $\mathrm{f}(\ln y)/y$, and deduce that the mode λ of Y satisfies $\mathrm{f}'(\ln\lambda) = \mathrm{f}(\ln\lambda)$.

(iii) Suppose now that $X \sim \mathrm{N}(\mu, \sigma^2)$, so that

$$\mathrm{f}(x) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}\,.$$

Explain why

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(x-\mu-\sigma^2)^2/(2\sigma^2)}\,\mathrm{d}x = 1$$

and hence show that $\mathrm{E}(Y) = \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}$.

(iv) Show that, when $X \sim \mathrm{N}(\mu, \sigma^2)$,

$$\lambda < y_m < \mathrm{E}(Y)\,.$$