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9. The volume is obtained as a volume of revolution $V = \int_x^R \pi(R^2 - t^2)dt$ which gives the result. Similarly, Newton's 2nd law gives $\frac{4}{3}\pi R^3 \rho_s \ddot{x} = V\rho g - \frac{4}{3}\pi R^3 \rho_s g$ which simplifies to the required result. Substituting $x = \frac{1}{2}R$ when $\ddot{x} = 0$ gives $\rho_s = \frac{5}{32}\rho$. Substituting $x = \frac{1}{2}R + y$ yields $\frac{5}{8}R^3 \ddot{y} = g\left(-\frac{9}{4}R^2 y + \frac{3}{2}Ry^2 + y^3\right)$, so for small y this approximates to SHM with period $\frac{\pi}{3}\sqrt{\frac{10R}{g}}$.