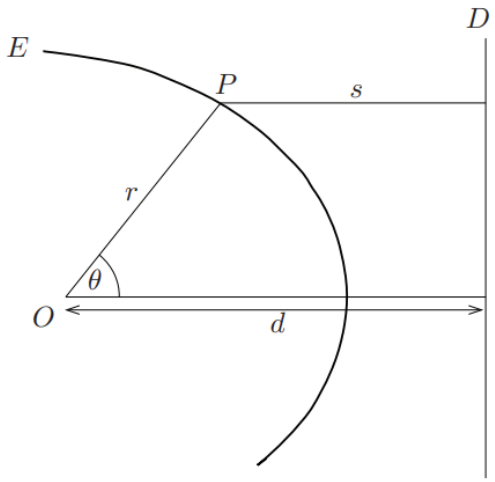
STEP III, 2013 , Q8

8 Evaluate $\displaystyle\sum_{r=0}^{n-1} \mathrm{e}^{2\mathrm{i}(\alpha+r\pi/n)}$ where α is a fixed angle and $n \geqslant 2$.

The fixed point O is a distance d from a fixed line D. For any point P, let s be the distance from P to D and let r be the distance from P to O. Write down an expression for s in terms of d, r and the angle θ, where θ is as shown in the diagram below.



The curve E shown in the diagram is such that, for any point P on E, the relation $r = ks$ holds, where k is a fixed number with $0 < k < 1$.

Each of the n lines $L_1, \ldots, L_n$ passes through O and the angle between adjacent lines is $\frac{\pi}{n}$. The line L_j $(j = 1, \ldots, n)$ intersects E in two points forming a chord of length l_j. Show that, for $n \geqslant 2$,

$$\sum_{j=1}^{n} \frac{1}{l_j} = \frac{(2-k^2)n}{4kd}.$$