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7. As $\frac{dE}{dx} = 2\,\frac{dy}{dx}\left(\frac{d^2y}{dx^2} + y^3\right)$ is zero for all x , $E(x)$ is constant, and $E(x) = \frac{1}{2}$ using the initial conditions. The deduction follows from the non-negativity of $\left(\frac{dy}{dx}\right)^2$. In part (ii), it can be shown that $\frac{dE}{dx} = -2x\left(\frac{dv}{dx}\right)^2 \leq 0$ for $x \geq 0$, and as initially $E(x) = \frac{10}{3}$, the deduction for $\cosh v(x)$ follows in the same way as that in part (i). In part (iii), the choice of $E(x)$ relies on $2\int(w\cosh w + 2\sinh w)dw$ so $E(x) = \left(\frac{dw}{dx}\right)^2 + 2\,(w\sinh w + \cosh w)$. Then

$\frac{dE}{dx} = -2\left(\frac{dw}{dx}\right)^2 (5\cosh x - 4\sinh x - 3) = -2\left(\frac{dw}{dx}\right)^2 \frac{e^{-x}}{2}\,(e^x - 3)^2$, and initially $E(x) = \frac{5}{2}$.

The final result can be deduced as in the previous parts, with the additional consideration that $w\sinh w \geq 0$.