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6. The opening result is the triangle inequality applied to OW, OZ, and WZ where OW and OZ are represented by the complex numbers w and z.

Part (i) relies on using $|z-w|^2 = (z-w)(z-w)^*$, $(z-w)^* = (z^* - w^*)$, $|zw| = |z|\,|w|$, and substituting $wz^* + zw^* = (E - 2|zw|)$. Having obtained the desired equation , the reality of E is apparent from the reality of the other terms and its non-negativity is obtained from the opening result of the question. Part (ii) relies on the same principles as part (i).

The inequality can be most easily obtained by squaring it, and substituting for both numerator and denominator on the left hand side using parts (i) and (ii), and algebraic rearrangement leads to $E(1-|z|^2)(1-|w|^2) \geq 0$ which is certainly true. The argument is fully reversible as $|z| > 1$, and $|w| > 1$, $|zw^*| > 1$, and so $1 - zw^* \neq 0$ so the division is permissible, and the square rooting of the inequality causes no problem as the quantities are positive. The working follows identically if $|z| < 1$, and $|w| < 1$.