STEP III, 2013 , Q6

6 Let z and w be complex numbers. Use a diagram to show that $|z - w| \leqslant |z| + |w|$.

For any complex numbers z and w, E is defined by

$$E = zw^* + z^*w + 2|zw|\,.$$

(i) Show that $|z - w|^2 = (|z| + |w|)^2 - E$, and deduce that E is real and non-negative.

(ii) Show that $|1 - zw^*|^2 = (1 + |zw|)^2 - E$.

Hence show that, if both $|z| > 1$ and $|w| > 1$, then

$$\frac{|z - w|}{|1 - zw^*|} \leqslant \frac{|z| + |w|}{1 + |zw|}\,.$$

Does this inequality also hold if both $|z| < 1$ and $|w| < 1$?