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5. Writing q^nN as $qq^{n-1}N$, and employing the permitted assumption, as p and q are coprime, p divides $q^{n-1}N$. Repetitions of this argument imply finally that p divides N . Letting $N = pQ_1$, $q^nQ_1 = p^{n-1}$. Continuing this argument similarly gives the result $N = kp^n$. As a consequence, $q^nk = 1$, and thus q and k must both be 1 . Thus if $\sqrt[n]{N} = \frac{p}{q}$ where p and q are coprime, it is rational and can be written in lowest terms, then $q^nN = p^n$ and so $q = 1$ and thus $\sqrt[n]{N}$ is an integer. Otherwise, $\sqrt[n]{N}$ cannot be written as $\frac{p}{q}$, that is, it is irrational.

For (ii), using the same logic as in part (i), as b^a divides a^ad^b , b^a divides d^b, so $d^b = kb^a$, for some k . Likewise, $a^a = k'c^b$, for some integer k' , and thus $k'k = 1$, so $k = k' = 1$, and $d^b = b^a$. If p is a prime factor of d , then p divides d^b , and so b^a too. Writing $b^a = bb^{a-1}$, using the logic of the very first part of the question, if p does not divide b , p divides b^{a-1}, and repetition of this argument leads to a contradiction. So p is a prime factor of b . p^{mb} and p^{na} is the highest power of p that divides $d^b = b^a$. So $mb = na$, and $b = \frac{na}{m}$. So p^n divides na , but as a and b are coprime, p^n divides n and thus $p^n \leq n$. By the given result, this means $p = 1$, and as b is only divisible by 1 , $b = 1$. If r is a positive rational $\frac{a}{b}$, such that $r^r = \frac{c}{d}$ is rational, then $a^ad^b = b^ac^b$ so $b = 1$ and r is a positive integer.