STEP III, 2013 , Q5

5 In this question, you may assume that, if a, b and c are positive integers such that a and b are coprime and a divides bc, then a divides c. (Two positive integers are said to be *coprime* if their highest common factor is 1.)

(i) Suppose that there are positive integers p, q, n and N such that p and q are coprime and $q^n N = p^n$. Show that $N = kp^n$ for some positive integer k and deduce the value of q.

Hence prove that, for any positive integers n and N, $\sqrt[n]{N}$ is either a positive integer or irrational.

(ii) Suppose that there are positive integers a, b, c and d such that a and b are coprime and c and d are coprime, and $a^a d^b = b^a c^b$. Prove that $d^b = b^a$ and deduce that, if p is a prime factor of d, then p is also a prime factor of b.

If p^m and p^n are the highest powers of the prime number p that divide d and b, respectively, express b in terms of a, m and n and hence show that $p^n \leqslant n$. Deduce the value of b. (You may assume that if $x > 0$ and $y \geqslant 2$ then $y^x > x$.)

Hence prove that, if r is a positive rational number such that r^r is rational, then r is a positive integer.