STEP III, 2013 , Q4

4 Show that $(z - \mathrm{e}^{\mathrm{i}\theta})(z - \mathrm{e}^{-\mathrm{i}\theta}) = z^2 - 2z\cos\theta + 1$.

Write down the $(2n)$th roots of -1 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\tfrac{(2k-1)\pi}{2n}\right) + 1 \right).$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = \mathrm{i}$ show that, when n is even,

$$\cos\left(\tfrac{\pi}{2n}\right)\cos\left(\tfrac{3\pi}{2n}\right)\cos\left(\tfrac{5\pi}{2n}\right)\cdots\cos\left(\tfrac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n}2^{1-n}.$$

(ii) Show that, when n is odd,

$$\cos^2\left(\tfrac{\pi}{2n}\right)\cos^2\left(\tfrac{3\pi}{2n}\right)\cos^2\left(\tfrac{5\pi}{2n}\right)\cdots\cos^2\left(\tfrac{(n-2)\pi}{2n}\right) = n2^{1-n}.$$

You may use without proof the fact that $1 + z^{2n} = (1+z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.