

STEP III, 2013, Q4

4 Show that $(z - e^{i\theta})(z - e^{-i\theta}) = z^2 - 2z \cos \theta + 1$.

Write down the $(2n)$ th roots of -1 in the form $e^{i\theta}$, where $-\pi < \theta \leq \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^n \left(z^2 - 2z \cos \left(\frac{(2k-1)\pi}{2n} \right) + 1 \right).$$

Here, n is a positive integer, and the \prod notation denotes the product.

(i) By substituting $z = i$ show that, when n is even,

$$\cos \left(\frac{\pi}{2n} \right) \cos \left(\frac{3\pi}{2n} \right) \cos \left(\frac{5\pi}{2n} \right) \cdots \cos \left(\frac{(2n-1)\pi}{2n} \right) = (-1)^{\frac{1}{2}n} 2^{1-n}.$$

(ii) Show that, when n is odd,

$$\cos^2 \left(\frac{\pi}{2n} \right) \cos^2 \left(\frac{3\pi}{2n} \right) \cos^2 \left(\frac{5\pi}{2n} \right) \cdots \cos^2 \left(\frac{(n-2)\pi}{2n} \right) = n 2^{1-n}.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.



NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to NextStepMaths.com