STEP III, 2013 , Q3 MS

3. The scalar product of p_i with $\sum p_r$, which is of course zero, can be expanded giving $p_i.p_i = 1$ and three products $p_i.p_j$ which are equal by symmetry, giving the required result. Expanding the expression suggested in (i), gives $\sum_{i=1}^{4}(p_i.p_i - 2x.p_i + x.x)$, which, bearing in mind that $p_i.p_i = 1$, $x.x = 1$, and that $x.\sum_{i=1}^{4} p_i = 0$, gives the correct result. Considering that $p_1.p_2 = -\frac{1}{3}$, $p_2.p_2 = 1$, and that a is positive, enables the given values to be found. Similarly $p_1.p_3 = -\frac{1}{3}$, $p_2.p_3 = -\frac{1}{3}$, and $p_3.p_3 = 1$ yields P_3, $P_4 = \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$. In (iii), using the logic of (i), $(XP_i)^4 = \left((p_i - x).(p_i - x)\right)^2 = 4(1 - x.p_i)^2$, as required. Expanding this, and using the coordinates of X and those of P_i that have been found,

$$\sum_{i=1}^{4}(XP_i)^4 = 16 + 4\left(z^2 + \left(\frac{2\sqrt{2}}{3}x - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x + \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x - \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2\right)$$

$= 16 + 4\left(\frac{4}{3}x^2 + \frac{4}{3}y^2 + \frac{4}{3}z^2\right) = \frac{64}{3}$ which is sufficient.