STEP III, 2013 , Q3

3 The four vertices P_i $(i = 1, 2, 3, 4)$ of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ $(i = 1, 2, 3, 4)$. Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \,.\, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i $(i = 1, 2, 3, 4)$.

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \,.\, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8\,.$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2\,.$$

By letting the coordinates of X be (x, y, z), show further that $\sum_{i=1}^{4} (XP_i)^4$ is independent of the position of X.