

### STEP III, 2013 , Q3

- 3 The four vertices  $P_i$  ( $i = 1, 2, 3, 4$ ) of a regular tetrahedron lie on the surface of a sphere with centre at  $O$  and of radius 1. The position vector of  $P_i$  with respect to  $O$  is  $\mathbf{p}_i$  ( $i = 1, 2, 3, 4$ ). Use the fact that  $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$  to show that  $\mathbf{p}_i \cdot \mathbf{p}_j = -\frac{1}{3}$  for  $i \neq j$ .

Let  $X$  be any point on the surface of the sphere, and let  $XP_i$  denote the length of the line joining  $X$  and  $P_i$  ( $i = 1, 2, 3, 4$ ).

- (i) By writing  $(XP_i)^2$  as  $(\mathbf{p}_i - \mathbf{x}) \cdot (\mathbf{p}_i - \mathbf{x})$ , where  $\mathbf{x}$  is the position vector of  $X$  with respect to  $O$ , show that

$$\sum_{i=1}^4 (XP_i)^2 = 8.$$

- (ii) Given that  $P_1$  has coordinates  $(0, 0, 1)$  and that the coordinates of  $P_2$  are of the form  $(a, 0, b)$ , where  $a > 0$ , show that  $a = 2\sqrt{2}/3$  and  $b = -1/3$ , and find the coordinates of  $P_3$  and  $P_4$ .

- (iii) Show that

$$\sum_{i=1}^4 (XP_i)^4 = 4 \sum_{i=1}^4 (1 - \mathbf{x} \cdot \mathbf{p}_i)^2.$$

By letting the coordinates of  $X$  be  $(x, y, z)$ , show further that  $\sum_{i=1}^4 (XP_i)^4$  is independent of the position of  $X$ .



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