STEP III, 2013 , Q2 MS

2. It is elegant to multiply by the denominator, then differentiate implicitly, and finally multiply by the same factor again to achieve the desired first result. The general result can be proved by then using induction, or by Leibnitz, if known. The general result can be used alongside the expression for y, and the first derived result with the substitution $x = 0$ to show that the general term of the Maclaurin series for even powers of x is zero, and for odd powers of x is $\frac{2^{2r}(r!)^2}{(2r+1)!}x^{2r+1}$. Thus, as $y = x + \frac{2^2}{3!}x^3 + \frac{4^2 2^2}{5!}x^5 + \cdots$ the required infinite sum is $\frac{y}{x}$ with $x = \frac{1}{2}$, that is $\frac{2\pi\sqrt{3}}{9}$.