STEP III, 2013 , Q1 MS

1. The first two results, whilst not necessarily included in current A2 specifications, are standard work. Applying them, $\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,dx = 2\int_0^1 \frac{1}{(1-a^2)+(t+a)^2}dt$, which can then be evaluated using a change of variable to give $\frac{2}{\sqrt{1-a^2}}\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)$. To simplify this to obtain the required result, $\tan\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)$ must be simplified using the relevant compound angle formula.

It is fairly straightforward to show that $I_{n+1} + 2I_n = \int_0^{\frac{1}{2}\pi} \sin^n x\,dx$, so applying this for $n = 2, 1, 0$ and applying the main result of the question to evaluate I_0 , gives $I_3 = \left(\frac{9}{4} - \frac{8\sqrt{3}}{9}\right)\pi - 2$