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13. integrating $0 \leq f(t) \leq M$ between limits of 0 and x gives the result of (a) (i), and integrating the left hand side by parts yields part (ii). As $kF(y)f(y)$ is a probability density function, $\int_0^1 k\,F(y)f(y)dy = 1$, which can be evaluated using the result of (a) (ii) with $2g(x) = k$ and so $k = 2$. $E(Y^n) = \int_0^1 y^n\, 2F(y)f(y)dy \leq \int_0^1 y^n\, 2Myf(y)dy = 2M\mu_{n+1}$ and using (a) (ii), $E(Y^n) = \int_0^1 y^n\, 2F(y)f(y)dy = 1 - n\int_0^1 y^{n-1}\left(F(y)\right)^2 dy$, as $\int_0^1 y^{n-1}\left(F(y)\right)^2 dy \leq \int_0^1 y^{n-1} My\, F(y)dy = M\int_0^1 y^n\, F(y)dy$, integration by parts gives $\int_0^1 y^n\, F(y)dy = \frac{1}{n+1} - \frac{1}{n+1}\mu_{n+1}$. Part (iii) is derived from part (ii) by rearranging

$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leq 2M\mu_{n+1}$ and making μ_{n+1} the subject, then translating $n+1$ to n.