Attempt this question independently, showing full and clear working.

STEP III 2013 Question 13

(a) The continuous random variable X satisfies $0 \leqslant X \leqslant 1$, and has probability density function $f(x)$ and cumulative distribution function $F(x)$. The greatest value of $f(x)$ is M, so that $0 \leqslant f(x) \leqslant M$.

(i) Show that $0 \leqslant F(x) \leqslant Mx$ for $0 \leqslant x \leqslant 1$.

(ii) For any function $g(x)$, show that

$$\int_0^1 2g(x)\,F(x)\,f(x)\,\mathrm{d}x = g(1) - \int_0^1 g'(x)\big(F(x)\big)^2\,\mathrm{d}x.$$

(b) The continuous random variable Y satisfies $0 \leqslant Y \leqslant 1$, and has probability density function $kF(y)f(y)$, where f and F are as above.

(i) Determine the value of the constant k.

(ii) Show that

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leqslant \mathrm{E}(Y^n) \leqslant 2M\mu_{n+1},$$

where $\mu_{n+1} = \mathrm{E}(X^{n+1})$ and $n \geqslant 0$.

(iii) Hence show that, for $n \geqslant 1$,

$$\mu_n \geqslant \frac{n}{(n+1)M} - \frac{n-1}{n+1}.$$

SCAN OR CLICK MARK SCHEME
SCAN OR CLICK EXAMINER'S COMMENTS