STEP III, 2013 , Q13

13 **(a)** The continuous random variable X satisfies $0 \leqslant X \leqslant 1$, and has probability density function $\mathrm{f}(x)$ and cumulative distribution function $\mathrm{F}(x)$. The greatest value of $\mathrm{f}(x)$ is M, so that $0 \leqslant \mathrm{f}(x) \leqslant M$.

(i) Show that $0 \leqslant \mathrm{F}(x) \leqslant Mx$ for $0 \leqslant x \leqslant 1$.

(ii) For any function $\mathrm{g}(x)$, show that

$$\int_0^1 2\mathrm{g}(x)\mathrm{F}(x)\mathrm{f}(x)\mathrm{d}x = \mathrm{g}(1) - \int_0^1 \mathrm{g}'(x)\big(\mathrm{F}(x)\big)^2\mathrm{d}x\,.$$

(b) The continuous random variable Y satisfies $0 \leqslant Y \leqslant 1$, and has probability density function $k\mathrm{F}(y)\mathrm{f}(y)$, where f and F are as above.

(i) Determine the value of the constant k.

(ii) Show that

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leqslant \mathrm{E}(Y^n) \leqslant 2M\mu_{n+1}\,,$$

where $\mu_{n+1} = \mathrm{E}(X^{n+1})$ and $n \geqslant 0$.

(iii) Hence show that, for $n \geqslant 1$,

$$\mu_n \geqslant \frac{n}{(n+1)M} - \frac{n-1}{n+1}\,.$$