STEP III, 2013 , Q12 MS

12. $P(X_1 = 1) = \frac{a}{n}$, so $E(X_1) = \frac{a}{n}$. There are $\frac{n!}{a!b!}$ arrangements of the As and Bs, and the number of arrangements with a B in the $(k-1)$ th place and an A in the k th place is $\frac{(n-2)!}{(a-1)!(b-1)!}$, so $P(X_k = 1) = \frac{ab}{n(n-1)}$ for $2 \le k \le n$, and $E(X_i) = \frac{ab}{n(n-1)}$ if $i \ne 1$. These combine to give $E(S)$ correctly.

$X_1X_j = 1$ only if the first letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A. This has probability $\frac{(n-3)!}{(a-2)!(b-1)!} / \frac{n!}{a!b!}$ giving $E(X_1X_j)$ correctly.

$X_iX_j = 1$ only if the $(i-1)$ th letter is a B, and the i th letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A which has probability $\frac{(n-4)!}{(a-2)!(b-2)!} / \frac{n!}{a!b!}$ so $E(X_iX_j) = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$, and thus $\sum_{j=i+2}^{n} E(X_iX_j) = (n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$ and so

$\sum_{i=2}^{n-2}\left(\sum_{j=i+2}^{n} E(X_iX_j)\right) = \sum_{i=2}^{n-2}\left((n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}\right)$ which yields the required result.

$S^2 = \sum_{i=1}^{n} {X_i}^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 2X_i X_j$ so $E(S^2) = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)}$ which can be used to obtain $Var(S)$ correctly.