

## STEP III, 2013 , Q12 MS

12.  $P(X_1 = 1) = \frac{a}{n}$ , so  $E(X_1) = \frac{a}{n}$ . There are  $\frac{n!}{a!b!}$  arrangements of the As and Bs, and the number of arrangements with a B in the  $(k - 1)$  th place and an A in the  $k$  th place is  $\frac{(n-2)!}{(a-1)!(b-1)!}$ , so  $P(X_k = 1) = \frac{ab}{n(n-1)}$  for  $2 \leq k \leq n$ , and  $E(X_i) = \frac{ab}{n(n-1)}$  if  $i \neq 1$ . These combine to give  $E(S)$  correctly.

$X_1X_j = 1$  only if the first letter is an A, the  $(j - 1)$  th letter is a B, and the  $j$  th letter is an A. This has probability  $\frac{(n-3)!}{(a-2)!(b-1)!} / \frac{n!}{a!b!}$  giving  $E(X_1X_j)$  correctly.

$X_iX_j = 1$  only if the  $(i - 1)$  th letter is a B, and the  $i$  th letter is an A, the  $(j - 1)$  th letter is a B, and the  $j$  th letter is an A which has probability  $\frac{(n-4)!}{(a-2)!(b-2)!} / \frac{n!}{a!b!}$  so  $E(X_iX_j) = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$ , and thus  $\sum_{j=i+2}^n E(X_iX_j) = (n - i - 1) \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$  and so

$$\sum_{i=2}^{n-2} \left( \sum_{j=i+2}^n E(X_iX_j) \right) = \sum_{i=2}^{n-2} \left( (n - i - 1) \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} \right) \text{ which yields the required result.}$$

$S^2 = \sum_{i=1}^n X_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^n 2X_i X_j$  so  $E(S^2) = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)}$  which can be used to obtain  $Var(S)$  correctly.



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