STEP III, 2013 , Q1

1 Given that $t = \tan \frac{1}{2}x$, show that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}(1+t^2)$ and $\sin x = \dfrac{2t}{1+t^2}$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,\mathrm{d}x = \frac{2}{\sqrt{1-a^2}} \arctan \frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0 < a < 1).$$

Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,\mathrm{d}x \qquad (n \geqslant 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate I_3.