STEP III, 2012 Q8

8 The sequence F_0, F_1, F_2, ... is defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geqslant 0$,

$$F_{n+2} = F_{n+1} + F_n \,.$$

(i) Show that $F_0F_3 - F_1F_2 = F_2F_5 - F_3F_4 \,.$

(ii) Find the values of $F_nF_{n+3} - F_{n+1}F_{n+2}$ in the two cases that arise.

(iii) Prove that, for $r = 1, 2, 3, \ldots,$

$$\arctan\left(\frac{1}{F_{2r}}\right) = \arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right)$$

and hence evaluate the following sum (which you may assume converges):

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right) \,.$$