STEP III, 2012 Q7

7 A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time t of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z)\,, \qquad \dot{z} = -\dot{y} - 3z\,,$$

where the dot denotes differentiation with respect to t.

Obtain a second order differential equation for y and hence derive the solution

$$y = Ae^{-t} + Be^{-6t}\,, \qquad z = \tfrac{1}{2}Ae^{-t} - 2Be^{-6t}\,,$$

where A and B are arbitrary constants.

(i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.

(ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where c is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.

(iii) Show that for $0 \leqslant t \leqslant 1$,

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t - n)\,,$$

provided c takes a particular value that you should find.