STEP III, 2012 Q5 MS

5. For non-integer rational points, it makes sense to use values of $\cos\theta$ and $\sin\theta$ based on Pythagorean triples such as 3, 4, 5 or 5, 12, 13, The technique for (i) (b) can be used for (ii) (a), merely by changing the value of m, whereas for (ii) (b), a slightly more involved expression is needed such as $x = a\cos\theta + b\sqrt{m}\sin\theta$, $y = a\sin\theta - b\sqrt{m}\cos\theta$. For (ii) (c), there are two alternatives that work sensibly, $= a\cosh\theta + \sqrt{m}\sinh\theta$, $y = a\sinh\theta + \sqrt{m}\cosh\theta$
or $x = a\sec\theta + \sqrt{m}\ \tan\theta$, $y = a\tan\theta + \sqrt{m}\ \sec\theta$.

A completely different approach for the last part of the question is to write
$x^2 - y^2 = (x+y)(x-y) = 7$ and to choose $x + y = a + b\sqrt{2}$ with nearly any choices of rational a and b possible. Then, as $x - y = \frac{7}{x+y}$, and numbers of the form $a + b\sqrt{2}$ are a field over the operations $\times$ and $+$, $x - y$ has to be of the correct form, and then solving for x and y, they likewise have to be of the required form.

Some possible solutions are

(i)(a) $(1,0)$ and $\left(\frac{3}{5},\frac{4}{5}\right)$, (b) $(1,1)$, $1+m$, and $\left(\frac{7}{5},\frac{1}{5}\right)$ (using $m = 1$, $\cos\theta = \frac{3}{5}$)

(ii)(a) $(1,\sqrt{2})$ and $\left(\frac{3}{5}+\frac{4}{5}\sqrt{2},\frac{4}{5}-\frac{3}{5}\sqrt{2}\right)$ (using $m = 2$, $\cos\theta = \frac{3}{5}$) , (b) $\left(\frac{9}{5}+\frac{4}{5}\sqrt{2},\frac{12}{5}-\frac{3}{5}\sqrt{2}\right)$ (using $a = 3$, $b = 1$, $\cos\theta = \frac{3}{5}$), (c) $\left(\frac{39}{5}+\frac{12}{5}\sqrt{2},\frac{36}{5}+\frac{13}{5}\sqrt{2}\right)$ (using $m = 2$, $a = 3$ and $\cosh\theta = \frac{13}{5}$, $\sinh\theta = \frac{12}{5}$ or $\sec\theta = \frac{13}{5}$, $\tan\theta = \frac{12}{5}$)